

FUNCTORIAL RECONSTRUCTION THEOREMS FOR STACKS

MAX LIEBLICH AND BRIAN OSSERMAN

ABSTRACT. We study the circumstances under which one can reconstruct a stack from its associated functor of isomorphism classes. This is possible surprisingly often: we show that many of the standard examples of moduli stacks are determined by their functors. Our methods seem to exhibit new anabelian-type phenomena, in the form of structures in the category of schemes that encode automorphism data in groupoids.

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1. INTRODUCTION

Let S be a scheme. Write \mathbf{Stacks}_S for the 1-category underlying the 2-category of fppf S -stacks with small fiber categories and \mathbf{Func}_S for the category of set-valued contravariant functors on the category of S -schemes.

There is a natural functor

$$F : \mathbf{Stacks}_S \rightarrow \mathbf{Func}_S$$

which sends a stack \mathcal{S} to its associated functor $F_{\mathcal{S}}$ of isomorphism classes (so that $F_{\mathcal{S}}(T)$ is the set of isomorphism classes of objects of \mathcal{S}_T).

Given a category \mathcal{C} , call a subclass P of $\mathbf{Obj} \mathcal{C}$ a *property* if it is closed under isomorphism. Given a property P of \mathcal{C} and a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, there is a *pushforward property* F_*P consisting of all objects of \mathcal{D} isomorphic to an object of $F(P)$. There is an associated category $\mathbf{hom}(\mathcal{C})$ consisting of diagrams in \mathcal{C} of the form $c \rightarrow d$; a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces a functor $\mathbf{hom}(\mathcal{C}) \rightarrow \mathbf{hom}(\mathcal{D})$, which we will also denote F (by abuse of notation). A property of $\mathbf{hom}(\mathcal{C})$ will also be called a *property of morphisms in \mathcal{C}* .

Definition. Given a subcategory \mathcal{C} of Stacks_S , a property P of \mathcal{C} (resp. $\text{hom}(\mathcal{C})$) is \mathcal{C} -*isonatural* if $P = (F|_{\mathcal{C}})^{-1}((F|_{\mathcal{C}})_*P)$.

It is straightforward that a property P is isonatural if and only if there is some property Q of Func_S (resp. $\text{hom}(\text{Func}_S)$) such that $P = F^{-1}(Q)$. The question which we address in this paper is the following.

Question. Which properties of Stacks_S (resp. $\text{hom}(\text{Stacks}_S)$) are isonatural (resp. $\text{hom}(F)$)?

Examples. (1) Given a stack \mathcal{X} , there is a property $[\mathcal{X}]$ consisting of the stacks isomorphic to \mathcal{X} . The statement that $[\mathcal{X}]$ is isonatural is the same as the statement that a stack \mathcal{Y} is isomorphic to \mathcal{X} if and only if $F_{\mathcal{Y}}$ is isomorphic to $F_{\mathcal{X}}$. In other words, \mathcal{X} is characterized up to 1-isomorphism by its associated functor of isomorphism classes. In this case, we will say “ \mathcal{X} is isonatural” in place of “ $[\mathcal{X}]$ is isonatural.”

(2) The subcategory of Deligne-Mumford stacks yields a property of Stacks_S . The statement that this property is isonatural is the same as the statement that if \mathcal{X} is Deligne-Mumford and $F_{\mathcal{X}}$ is isomorphic to $F_{\mathcal{Y}}$ then \mathcal{Y} is Deligne-Mumford. In other words, there is a functorial criterion for a stack to be Deligne-Mumford.

(3) The subcategory of $\text{hom}(\text{Stacks}_S)$ parametrizing representable, smooth, etc., morphisms $\mathcal{X} \rightarrow \mathcal{Y}$ defines a property. Isonaturality means that this a morphism $\mathcal{X} \rightarrow \mathcal{Y}$ has this property if and only if the induced map $F_{\mathcal{X}} \rightarrow F_{\mathcal{Y}}$ has some other property (in the formal sense), which one would ideally like to describe.

For all of the properties and stacks that we can prove are isonatural, we explicitly describe the corresponding properties of the associated functors (resp. the morphisms of associated functors). This constructive aspect of our proofs requires that we restrict our attention to a particular subcategory \mathcal{Q} of Stacks_S , which we call *quasi-algebraic stacks*. The reader is referred to Definition 1.1.3 for a glimpse of this subcategory.

Convention. In the rest of this paper, “isonatural” will mean “ \mathcal{Q} -isonatural.”

At first glance, it may seem that there is no hope of recovering the automorphism data contained in the stack after passing to $F_{\mathcal{X}}$, but it turns out that this is not the case. Theorem 1.2.1 asserts that many classical moduli stacks are in fact isonatural. More generally, Theorem 3.1.8 offers substantial evidence for a positive answer to the following question.

Question. Are all quasi-algebraic stacks isonatural?

As we show in Section 3.6, this is not a purely abstract statement about stacks on sites (even if one considers only stacks with representable diagonals), as there are many examples of stacks on the small étale site of a field or geometrically unibranch scheme which are not isonatural.

To the reader used to thinking about the theory of moduli (and who has learned the standard phylogeny which proceeds from functors to sheaves to stacks), the results we present here may seem surprising. If the information contained in the category fibered in groupoids is not contained in the isomorphism data (as we are taught), then where is it? The answer, of course, lies in descent theory, which creates a tight relationship between the sets of isomorphism classes of objects and their automorphism groups.

The situation is evocative of anabelian geometry. In anabelian theory, and its subsequent extensions by Mochizuki, auxiliary categories – the category of finite étale covers [19] [23], or the slice category of (log) schemes [20] – can be shown to determine a scheme, typically by explicit reconstructive arguments. Our results show, roughly, that structures in the category of S -schemes can serve to reconstruct groupoids from the associated “coarse” data contained in functors.

A basic example of this kind of structure arises in studying the classifying stack BG over an algebraically closed field k , where G is a finite group. As we show in Proposition 3.3.6, there is a faithful functor β from the category of finite groups to the category of pointed k -schemes such that for each finite group G there is a natural isomorphism $\pi_1(\beta(G)) \xrightarrow{\sim} G$. In this way, one can recover the functor of points of G in the category of bands over a point, which is enough to recover BG . In classical language,

if k is an algebraically closed field and G is a finite group, the functor $X \mapsto H^1(X, G)$ on the category of k -schemes determines G up to unique outer isomorphism.

The phenomena we describe strike us as pedagogically useful: even if one is primarily concerned with the isomorphism classes of a given moduli problem, the automorphism information in the stack is nonetheless already determined by the crude functorial description.

1.1. Notation and basic definitions. Prior to summarizing our results, we set the notation and terminology used throughout the paper.

We fix a quasi-separated base scheme S throughout. (This assumption is also hidden in [17] on page x; since this is the standard reference on the subject, we will also make blanket quasi-separation hypotheses.)

Given a category \mathcal{C} and object $T \in \mathcal{C}$, the slice category of \mathcal{C} over T will be denoted $T\text{-}\mathcal{C}$. A functor $F : \mathcal{C}^\circ \rightarrow \mathbf{Set}$ induces a functor $F|_T : (T\text{-}\mathcal{C})^\circ \rightarrow \mathbf{Set}$.

We assume throughout that all stacks have small fiber categories. Grothendieck's theory of universes can be used to see that this is a harmless assumption in practice.

An *open substack* of a stack \mathcal{X} is an equivalence class of morphisms $\mathcal{U} \rightarrow \mathcal{X}$ which are representable by open immersions.

Definition 1.1.1. A stack is *abelian* if it has abelian inertia stack.

Following [17], all Artin stacks will be assumed to have quasi-compact (and hence finite type) diagonals. The notion of quasi-algebraic stack involves an infinitesimal condition, for which we recall the following.

Lemma 1.1.2. *Given $U_0 = \operatorname{Spec} A_0 \rightarrow U = \operatorname{Spec} A$ a nilpotent closed immersion of affine S -schemes, and a morphism $U_0 \rightarrow Z$ with $Z = \operatorname{Spec} B$ another affine S -scheme, the pushout $U \amalg_{U_0} Z$ exists in the category of S -schemes, and is given by $\operatorname{Spec}(A \times_{A_0} B)$.*

Proof. One checks easily that the map $Z \rightarrow \operatorname{Spec}(A \times_{A_0} B)$ is a homeomorphism, and it is then not difficult to verify that $\operatorname{Spec}(A \times_{A_0} B)$ is indeed the pushout in the category of (possibly non-affine) schemes. \square

Definition 1.1.3. We call an S -stack \mathcal{X} *quasi-algebraic* if it satisfies the following three conditions.

- (1) The diagonal $\mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is representable by separated quasi-compact morphisms of algebraic spaces.
- (2) The inertia stack of \mathcal{X} is locally of finite presentation over \mathcal{X} .
- (3) Given affine S -schemes U_0, U, Z with $U_0 \hookrightarrow U$ a nilpotent closed immersion and $U_0 \rightarrow Z$ an arbitrary morphism, the induced functor between fiber categories

$$\mathcal{X}_U \amalg_{\mathcal{X}_{U_0}} \mathcal{X}_Z \rightarrow \mathcal{X}_U \times_{\mathcal{X}_{U_0}} \mathcal{X}_Z$$

is an equivalence.

Remark 1.1.4. Conditions (1) and (3) are always satisfied by any Artin stack (see e.g. Lemma 1.4.4 of [21] for the latter). Condition (2) is satisfied by any algebraic space, and any locally Noetherian Artin stack, and more generally any Artin stack for which the diagonal is locally of finite presentation, but not for a general Artin stack. Thus, our implicit hypothesis that all Artin and Deligne-Mumford stacks are quasi-algebraic is a non-vacuous restriction.

For a simple example of this, consider the action of $\mathbf{Z}/2\mathbf{Z}$ on $k[x_1, x_2, \dots]$ in which 1 acts by multiplication by -1 on each variable. The quotient stack is an Artin stack, but is not locally Noetherian, and does not satisfy (2). Indeed, when pulled back to $\operatorname{Spec} k[x_1, x_2, \dots]$, the inertia stack is given by the extension by 0 of the group scheme $\mathbf{Z}/2\mathbf{Z}$ supported at the “origin,” and is therefore not locally of finite presentation.

Each of conditions (2) and (3) only arise at a single point in our argument, in recognizing morphisms which are locally of finite presentation and smooth/étale respectively, but these are crucial because they combine to show we can recognize smooth covers by schemes on the level of functors, which then leads to a plethora of additional recognition results.

We will without further comment assume that all of our Artin (and Deligne-Mumford) stacks satisfy condition (2) as well, so that they are quasi-algebraic.

1.2. Summary of results. In Section 2, we will analyze properties of morphisms of Artin stacks, as well as absolute properties of stacks, showing that nearly all of the standard properties can be tested on the level of functors. As detailed in Section 2.1, nearly all standard properties of morphisms and of stacks are isonatural. Most of these results follow after we show that we can recognize smooth covers of a quasi-algebraic stack by a scheme, although additional argument is required for quasi-compact, separated, and proper morphisms. We also show that we can test non-triviality of stabilizer groups from the functor, and can in fact recover the groups whenever they are abelian.

In Section 3, we analyze a number of specific classes of stacks, such as Artin stacks having an open dense substack with trivial stabilizer, classifying stacks for finite and abelian groups, and certain gerbes. We use specific categorical construction in each case to show that within each class, a stack can be reconstructed from its functor, and we then apply the results of Section 2 to show that we can also tell on the level of functors whether a quasi-algebraic stack is in any of the classes in question. We thus conclude that any stack in any of the classes is isonatural. For detailed statements, see Section 3.1.

The following theorem is a corollary of our main results, and shows that the most common moduli stacks are in fact all isonatural.

Theorem 1.2.1. *Any base change of any open substack of any of the following stacks is isonatural.*

- (1) *The stack $\overline{\mathcal{M}}_{g,n}$ for all $g \geq 2$ and $n \geq 0$, as a \mathbf{Z} -stack.*
- (2) *The Picard stack of a projective scheme flat and of finite presentation which is cohomologically flat in degree 0 over an algebraic space with quasi-compact connected components.*
- (3) *The stack of stable vector bundles on a projective scheme flat and of finite presentation which is cohomologically flat in degree 0 over an algebraic space with quasi-compact connected components.*
- (4) *The stack of stable vector bundles of rank n and fixed determinant on a smooth proper curve over any normal quasi-projective $\mathbf{Z}[1/n]$ -scheme.*
- (5) *The stack of stable coherent sheaves of rank n with fixed determinant and sufficiently large discriminant on a smooth projective surface over any normal quasi-projective $\mathbf{Z}[1/n]$ -scheme.*
- (6) *The stack of n th roots of an invertible sheaf \mathcal{L} on a regular algebraic space X .*

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2. ISONATURAL PROPERTIES

In this section, we prove that nearly all of the usual properties of quasi-algebraic stacks, and of morphisms of Artin stacks, are isonatural. We impose additional hypotheses for separateness and properness, but otherwise our results are completely general.

2.1. Summary of results. Suppose we are given a 1-morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks over a base scheme S with induced morphism $F_f : F_{\mathcal{X}} \rightarrow F_{\mathcal{Y}}$.

We remind the reader that a *trait* is the spectrum of a complete discrete valuation ring.

Theorem 2.1.1. *The following properties of representable morphisms of quasi-algebraic stacks are isonatural:*

- (1) *locally of finite presentation;*
- (2) *surjective;*
- (3) *smooth, assuming the source is a scheme;*
- (4) *unramified, assuming the source is a scheme;*
- (5) *étale, assuming the source is a scheme.*

Theorem 2.1.2. *The following properties of morphisms of Artin stacks are isonatural:*

- (1) *locally of finite presentation;*

- (2) *locally of finite type*;
- (3) *surjective*;
- (4) *smooth*;
- (5) *flat*;
- (6) *quasi-compact*;
- (7) *separated and locally of finite type, assuming the target is locally Noetherian*;
- (8) *proper, assuming the target is locally Noetherian and either is abelian or has proper inertia*.

We will use these results, and in particular our ability to recognize smooth covers by a scheme, to show further that a number of absolute properties of stacks are isonatural.

Corollary 2.1.3. *The following properties of quasi-algebraic stacks are isonatural:*

- (1) *Artin*;
- (2) *Deligne-Mumford*;
- (3) *gerbe (over an algebraic space)*;
- (4) *locally Noetherian*;
- (5) *normal*;
- (6) *reduced*;
- (7) *regular*;
- (8) *quasi-compact*;
- (9) *proper inertia*.

Note that for locally Noetherian, normal, etc., we are assuming that the stack in question is an Artin stack, because this is the only context in which the properties are defined. However, we can also test whether or not a stack is Artin on the level of functors.

Finally, in Section 2.7 we show the following.

Theorem 2.1.4. *Let \mathcal{X} be a quasi-algebraic stack, and $\eta \in \mathcal{X}_T$ for some scheme T . Then the following can be recovered from the functor $F_{\mathcal{X}}$:*

- (1) *whether or not $\text{Aut}(\eta)$ is abelian*;
- (2) *$\text{Aut}(\eta)$ itself, when it is abelian*.

Moreover, given a morphism $T' \rightarrow T$, if $\text{Aut}(\eta)$ and $\text{Aut}(\eta|_{T'})$ are both abelian, the natural restriction map $\text{Aut}(\eta) \rightarrow \text{Aut}(\eta|_{T'})$ can also be recovered from $F_{\mathcal{X}}$.

Since the trivial group is abelian, it follows from Theorem 2.1.4 that triviality of $\text{Aut}(\eta)$ is also determined by $F_{\mathcal{X}}$. We therefore immediately conclude (see Corollary 2.7.16 for a more precise version).

Corollary 2.1.5. *Algebraic spaces are isonatural among quasi-algebraic stacks.*

2.2. Background on categories. We begin by reminding the reader of several basic notions from the theory of 2-categories (by which we mean categories enriched over groupoids). The reader afraid of arbitrary 2-categories can simply think about the 2-category of categories: the objects are categories and the groupoid of morphisms between two objects is the groupoid of functors (with 2-isomorphisms given by natural isomorphisms). In what follows, we will write \mathcal{C} for a fixed 2-category.

Definition 2.2.1. A 2-commutative diagram in \mathcal{C} is a commutative diagram in the underlying 1-category $\overline{\mathcal{C}}$ along with a collection of 2-isomorphisms determined as follows:

- (1) for any two objects A and B in the diagram and any two arrows α and β between A and B arising from compositions of arrows in the diagram, there is a 2-isomorphism $\gamma_{\alpha,\beta} : \alpha \rightarrow \beta$;
- (2) for any three paths α, β, δ from A to B , we have $\gamma_{\beta,\delta}\gamma_{\alpha,\beta} = \gamma_{\alpha,\delta}$;
- (3) given two paths $\alpha, \beta : A \rightarrow B$ and a path $\delta : B \rightarrow C$, we have $\delta(\gamma_{\alpha,\beta}) = \gamma_{\delta\alpha,\delta\beta}$, where the left-hand side arises from the composition functor on hom-groupoids.

In other words, the commuting of 1-morphisms is mediated by 2-morphisms, and any conceivable commutation relation among the 2-morphisms is assumed to be compatible. Commutative diagrams in this section are always assumed to be 2-commutative; we will suppress the 2-morphisms from the notation.

Definition 2.2.2. A 2-Cartesian diagram in \mathcal{C} is a square of 1-morphisms

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ f \downarrow & & \downarrow g \\ C & \xrightarrow{h} & D \end{array}$$

along with an isomorphism $\epsilon : g \circ e \xrightarrow{\sim} h \circ f$, such that for all objects E of \mathcal{C} , the functor

$$\mathrm{Hom}(E, A) \rightarrow \mathrm{Hom}(E, C) \times_{\mathrm{Hom}(E, D)} \mathrm{Hom}(E, B)$$

is an equivalence of groupoids. Here the right-hand side is the usual fibered product of groupoids (see Example 2.2.3 below), and ϵ is used to compare the compositions $E \rightarrow B \rightarrow D$ and $E \rightarrow C \rightarrow D$ coming from $E \rightarrow A$.

Example 2.2.3. Suppose

$$\begin{array}{ccc} & B & \\ & \downarrow g & \\ C & \xrightarrow{h} & D \end{array}$$

is a pair of functors between (small) categories. Define $B \times_D C$ to be the category of triples (b, c, ϵ) , where b is an object of B , c is an object of C , and $\epsilon : g(b) \xrightarrow{\sim} h(c)$ is an isomorphism. Then the resulting diagram

$$\begin{array}{ccc} B \times_D C & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is 2-Cartesian.

Lemma 2.2.4. Consider a 2-commutative diagram in \mathcal{C} of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' \end{array}$$

If the lower square is 2-Cartesian then the upper square is 2-Cartesian if and only if the outer square is 2-Cartesian.

Proof. Let ϵ, δ, η be the isomorphisms associated to the upper, lower, and outer squares respectively by the 2-commutativity of the diagram. Since the lower square is 2-Cartesian, the functor

$$\mathrm{Hom}(E, A') \rightarrow \mathrm{Hom}(E, B') \times_{\mathrm{Hom}(E, B'')} \mathrm{Hom}(E, A'')$$

induced by δ is an equivalence, which leads to an equivalence

$$\begin{array}{c} \mathrm{Hom}(E, B) \times_{\mathrm{Hom}(E, B')} \mathrm{Hom}(E, A') \\ \downarrow \\ \mathrm{Hom}(E, B) \times_{\mathrm{Hom}(E, B')} (\mathrm{Hom}(E, B') \times_{\mathrm{Hom}(E, B'')} \mathrm{Hom}(E, A'')). \end{array}$$

Thus, we see that the natural functor

$$\mathrm{Hom}(E, B) \times_{\mathrm{Hom}(E, B')} \mathrm{Hom}(E, A') \rightarrow \mathrm{Hom}(E, B) \times_{\mathrm{Hom}(E, B'')} \mathrm{Hom}(E, A'')$$

induced by δ is an equivalence of groupoids. Moreover, by 2-commutativity we find that the natural functor $\mathrm{Hom}(E, A) \rightarrow \mathrm{Hom}(E, B) \times_{\mathrm{Hom}(E, B'')} \mathrm{Hom}(E, A'')$ induced by η factors as the composition of the functor $\mathrm{Hom}(E, A) \rightarrow \mathrm{Hom}(E, B) \times_{\mathrm{Hom}(E, B')} \mathrm{Hom}(E, A')$ induced by ϵ with the above equivalence. It follows that the top square satisfies the 2-Cartesian property if and only if the outer square does. \square

We next describe a model for certain homotopy colimits which will arise in our study of finiteness properties. Let $\mathcal{X} \rightarrow S$ be a category fibered in groupoids over $S\text{-Sch}$. The following lemma is well known.

Lemma 2.2.5. *There is a functorial pair $(\mathcal{X}^{\mathrm{split}}, \sigma : \mathcal{X}^{\mathrm{split}} \rightarrow \mathcal{X})$ consisting of a split S -groupoid and a 1-isomorphism σ of S -groupoids, such that for any 1-morphism $\mathcal{X} \rightarrow \mathcal{Y}$, the diagram*

$$\begin{array}{ccc} \mathcal{X}^{\mathrm{split}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}^{\mathrm{split}} & \longrightarrow & \mathcal{Y} \end{array}$$

strictly commutes (in the sense that the two compositions are equal as functors $\mathcal{X}^{\mathrm{split}} \rightarrow \mathcal{Y}$).

Proof. Given \mathcal{X} , define $\mathcal{X}^{\mathrm{split}}$ as the groupoid whose fiber category over $T \rightarrow S$ is the groupoid of maps $\mathrm{Hom}_S(T, \mathcal{X})$ (where T denotes by abuse of notation the canonical discrete groupoid associated to T). Composition of morphisms gives $\mathcal{X}^{\mathrm{split}}$ the structure of a split groupoid (i.e., a functor from $S\text{-Sch}^\circ$ to the category of groupoids which satisfies the descent condition). Moreover, evaluation on the identity yields a natural map $\mathcal{X}^{\mathrm{split}} \rightarrow \mathcal{X}$ which is a 1-isomorphism since any arrow in a category fibered in groupoids is Cartesian. Functoriality is clear from the construction. \square

We recall the naïve notion of colimit in the category of categories. Suppose we have a filtering directed system of categories and functors (\mathcal{C}_i) ; here we assume that the functors are strictly associative, so that we obtain a functor to the category of categories. Given also $x \in \mathrm{Obj} \mathcal{C}_i$, write $x|_{\mathcal{C}_k}$ for the image of x under the map $\mathcal{C}_i \rightarrow \mathcal{C}_k$ for any $k \geq i$. There is a colimit $\varinjlim \mathcal{C}_i$ in the category of categories defined as follows: the objects of $\varinjlim \mathcal{C}_i$ are given by the disjoint union $\coprod \mathrm{Obj} \mathcal{C}_i$, and, if $x \in \mathrm{Obj} \mathcal{C}_i$ and $y \in \mathrm{Obj} \mathcal{C}_j$, the morphisms $\mathrm{Hom}_{\varinjlim \mathcal{C}_i}(x, y)$ are given by $\varinjlim_{k \geq i, j} \mathrm{Hom}_{\mathcal{C}_k}(x|_{\mathcal{C}_k}, y|_{\mathcal{C}_k})$.

Now let R_i be a filtering directed system of S -rings, and write $R = \varinjlim R_i$. If \mathcal{X} is an S -groupoid, we observe that $\mathcal{X}_{R_i}^{\mathrm{split}}$ forms a filtering direct system of categories.

Definition 2.2.6. The colimit of \mathcal{X} over (R_i) , denoted $\varinjlim \mathcal{X}_{R_i}$, is the naïve colimit $\varinjlim \mathcal{X}_{R_i}^{\mathrm{split}}$.

The definition makes it clear that there is a natural morphism $\varinjlim \mathcal{X}_{R_i} \rightarrow \mathcal{X}_R^{\mathrm{split}}$, which, composed with σ_R yields a natural 1-morphism $\varinjlim \mathcal{X}_{R_i} \rightarrow \mathcal{X}_R$. Thus, given a 1-morphism $\mathcal{X} \rightarrow \mathcal{Y}$, there thus results a diagram

$$\begin{array}{ccc} \varinjlim \mathcal{X}_{R_i} & \longrightarrow & \mathcal{X}_R \\ \downarrow & & \downarrow \\ \varinjlim \mathcal{Y}_{R_i} & \longrightarrow & \mathcal{Y}_R \end{array}$$

which is strictly commutative.

2.3. Finiteness properties. We will show in Proposition 2.3.10 that the finiteness hypothesis on the inertia stack of a quasi-algebraic stack implies that for morphisms of quasi-algebraic stacks, being locally of finite presentation is isonatural.

Definition 2.3.1. Given a scheme S , a morphism of S -groupoids $A \rightarrow B$ is *locally of finite presentation* if, for all filtering directed systems R_i of S -rings, the diagram

$$\begin{array}{ccc} \varinjlim A_{R_i} & \longrightarrow & A_{\varinjlim R_i} \\ \downarrow & & \downarrow \\ \varinjlim B_{R_i} & \longrightarrow & B_{\varinjlim R_i} \end{array}$$

is 2-Cartesian.

Remark 2.3.2. Any functor has an associated (discrete) groupoid. Applying Definition 2.3.1 in the case that A and B arise as the groupoids associated to functors yields the usual definition of local finite presentation for natural transformations between functors. Furthermore, when A and B are the functors of points of S -schemes, this definition coincides with the standard definition for schemes, by Proposition 8.14.2 of [9]. More generally, if A and B are Artin stacks, our definition agrees with the usual one, by Proposition 4.15(i) of [17]. We will also verify in Lemma 2.3.5 below that our definition agrees with that of [17] in the case of representable morphisms.

From Lemma 2.2.4, we formally conclude the following.

Corollary 2.3.3. *Let S be a scheme, and*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

morphisms of fibered S -categories, with g locally of finite presentation. Then f is locally of finite presentation if and only if $g \circ f$ is locally of finite presentation.

Lemma 2.3.4. *Let $A \rightarrow B$ and $C \rightarrow B$ be morphisms of S -groupoids. If $A \rightarrow B$ is locally of finite presentation then the pullback $A \times_B C \rightarrow C$ is locally of finite presentation.*

Proof. Let R_i be a filtering directed system of S -rings. Consider the diagram

$$\begin{array}{ccccc} (A \times_B C)_{\varinjlim R_i} & \xrightarrow{\hspace{10em}} & A_{\varinjlim R_i} & & \\ \downarrow & \swarrow & \downarrow & \nearrow & \downarrow \\ & \varinjlim (A \times_B C)_{R_i} \longrightarrow \varinjlim A_{R_i} & & & \\ & \downarrow & \downarrow & & \\ & \varinjlim C_{R_i} \longrightarrow \varinjlim B_{R_i} & & & \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ C_{\varinjlim R_i} & \xrightarrow{\hspace{10em}} & B_{\varinjlim R_i} & & \end{array}$$

The right “square” of the diagram is 2-Cartesian by hypothesis and the outer square is 2-Cartesian by definition of the fiber product. The inner square is 2-Cartesian since colimits of 2-Cartesian squares are 2-Cartesian. Applying Lemma 2.2.4, we see that the left square is 2-Cartesian, as desired. \square

Definition 3.10.1 of [17] defines a representable morphism of stacks $\mathcal{X} \rightarrow \mathcal{Y}$ to be locally of finite presentation if for all $U \rightarrow \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} U \rightarrow U$ is locally of finite presentation. As we show in the following lemma, this is actually no different from our definition above.

Lemma 2.3.5. *A representable morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks is locally of finite presentation if and only if for all schemes $U \rightarrow \mathcal{Y}$, the algebraic space $\mathcal{X} \times_{\mathcal{Y}} U \rightarrow U$ is locally of finite presentation.*

Proof. If f is locally of finite presentation and $U \rightarrow \mathcal{Y}$ is an object, then the morphism $\mathcal{X} \times_{\mathcal{Y}} U \rightarrow U$ is locally of finite presentation by Lemma 2.3.4 and Remark 2.3.2.

Now assume that $f : \mathcal{X} \times_{\mathcal{Y}} U \rightarrow U$ is locally of finite presentation for all objects $U \rightarrow \mathcal{Y}$, and let R_i be a filtering directed system of S -rings. By Proposition I.8.1.6 of [2], there is a cofinal subsystem

whose indexing category is a partially ordered set with an initial element. This yields R_0 in the system which maps uniquely to every R_i so that the system becomes a system of R_0 -algebras.

Let α be an object of $\mathcal{X}_{\varinjlim R_i}$ with image β in $\mathcal{Y}_{\varinjlim R_i}$. Moreover, suppose $b \in \mathcal{Y}_{R_1}$ is an object and $\phi : b \rightarrow \beta$ is an isomorphism in $\mathcal{Y}_{\varinjlim R_i}$. We may clearly suppose that $R_1 = R_0$. We wish to show that the triple (α, b, ϕ) arises from an object γ_i of \mathcal{X}_{R_i} for some i , and that any two such objects γ_i and γ_j become uniquely isomorphic in \mathcal{X}_{R_k} for some k larger than i and j .

Let $U = \text{Spec } R_0$. The system R_i becomes a system of U -rings, and the relative finite presentation condition tells us that the diagram

$$\begin{array}{ccc} \varinjlim (\mathcal{X} \times_{\mathcal{Y}} U)_{R_i} & \longrightarrow & (\mathcal{X} \times_{\mathcal{Y}} U)_{\varinjlim R_i} \\ \downarrow & & \downarrow \\ \varinjlim U_{R_i} & \longrightarrow & U_{\varinjlim R_i} \end{array}$$

is 2-Cartesian. Using the compatibility of colimits with fibered product, we can rewrite the diagram as

$$(1) \quad \begin{array}{ccc} \varinjlim \mathcal{X}_{R_i} \times \varinjlim \mathcal{Y}_{R_i} \varinjlim U_{R_i} & \longrightarrow & \mathcal{X}_{\varinjlim R_i} \times \mathcal{Y}_{\varinjlim R_i} U_{\varinjlim R_i} \\ \downarrow & & \downarrow \\ \varinjlim U_{R_i} & \longrightarrow & U_{\varinjlim R_i}. \end{array}$$

The system yields a canonical element $a \in U_{\varinjlim R_i}$. The triple (α, b, ϕ) yields an object of the upper right category of diagram (1) mapping to a . On the other hand, a is the image of any of the structure morphisms occurring in $\varinjlim U_{R_i}$. Since (1) is 2-Cartesian, we see that there is a γ_i mapping to (α, b, ϕ) , and that any two γ_i and γ_j become uniquely isomorphic in \mathcal{X}_{R_k} for large enough k , as required. \square

Notation 2.3.6. Given a (small) category C , we denote by F_C the set of isomorphism classes of objects in C . We also denote by I_C the category whose objects are pairs (η, φ) with $\eta \in C$, $\varphi \in \text{Aut}(C)$, and whose morphisms are morphisms $f : \eta \rightarrow \eta'$ in C with $f \circ \varphi = \varphi' \circ f$.

Lemma 2.3.7. *Given a directed system of categories A_i , the natural map $\varinjlim F_{A_i} \rightarrow F_{\varinjlim A_i}$ is a bijection. The natural morphism $\varinjlim I_{A_i} \rightarrow I_{\varinjlim A_i}$ is an equivalence.*

Proof. The first assertion follows easily from the definition of a colimit of categories. The second is slightly more involved, but still routine. \square

Proposition 2.3.8. *For any quasi-algebraic S -stack \mathcal{X} , the natural map $\mathcal{X} \rightarrow F_{\mathcal{X}}$ is locally of finite presentation.*

Proof. We will show that this follows from the hypothesis that the inertia stack of \mathcal{X} is finitely presented over \mathcal{X} . We need to see that for all filtering directed systems R_i of S -rings, the diagram

$$\begin{array}{ccc} \varinjlim \mathcal{X}_{R_i} & \longrightarrow & \mathcal{X}_{\varinjlim R_i} \\ \downarrow & & \downarrow \\ \varinjlim F_{\mathcal{X}}(R_i) & \longrightarrow & F_{\mathcal{X}}(\varinjlim R_i) \end{array}$$

is 2-Cartesian. By the first part of Lemma 2.3.7, we may replace $\varinjlim F_{\mathcal{X}}(R_i)$ by $F_{\varinjlim \mathcal{X}_{R_i}}$. We then need to check that the morphism

$$n : \varinjlim \mathcal{X}_{R_i} \rightarrow \mathcal{X}_{\varinjlim R_i} \times_{F_{\mathcal{X}}(\varinjlim R_i)} F_{\varinjlim \mathcal{X}_{R_i}}$$

is essentially surjective and fully faithful.

An object of $\mathcal{X}_{\varinjlim R_i} \times_{F_{\mathcal{X}}(\varinjlim R_i)} F_{\varinjlim \mathcal{X}_{R_i}}$ is a pair $(\eta, \bar{\mu})$ with $\eta \in \mathcal{X}_{\varinjlim R_i}$ and $\bar{\mu} \in F_{\varinjlim \mathcal{X}_{R_i}}$, both mapping to the same element in $F_{\mathcal{X}}(\varinjlim R_i)$. But if $\mu \in \varinjlim \mathcal{X}_{R_i}$ is an object representing $\bar{\mu}$, we check easily that that μ maps to an object isomorphic to $(\eta, \bar{\mu})$, proving essential surjectivity.

We claim that the full faithfulness of n follows from the hypothesis that the inertia stack of \mathcal{X} is locally of finite presentation over \mathcal{X} . We note that a morphism $(\eta, \bar{\mu}) \rightarrow (\eta', \bar{\mu}')$ in $\mathcal{X}_{\varinjlim R_i} \times_{F_{\mathcal{X}}(\varinjlim R_i)} F_{\varinjlim \mathcal{X}_{R_i}}$ is simply a morphism $\eta \rightarrow \eta'$, together with the requirement that $\bar{\mu} = \bar{\mu}'$. If $(\eta, \bar{\mu})$ and $(\eta', \bar{\mu}')$ are the images of $\mu, \mu' \in \varinjlim \mathcal{X}_{R_i}$, we therefore need to check that under the hypothesis that μ, μ' are isomorphic, the (iso)morphisms $\mu \rightarrow \mu'$ are in bijection with (iso)morphisms $\eta \rightarrow \eta'$. Fixing a choice of isomorphism $\mu \rightarrow \mu'$, it is therefore enough to see that $\text{Aut}(\mu)$ is in bijection with $\text{Aut}(\eta)$.

This last assertion is precisely what is given by the hypothesis that the inertia stack of \mathcal{X} is locally of finite presentation over \mathcal{X} : in the 2-Cartesian diagram

$$\begin{array}{ccc} \varinjlim \mathcal{I}(\mathcal{X})_{R_i} & \longrightarrow & \mathcal{I}(\mathcal{X})_{\varinjlim R_i} \\ \downarrow & & \downarrow \\ \varinjlim \mathcal{X}_{R_i} & \longrightarrow & \mathcal{X}_{\varinjlim R_i}, \end{array}$$

after applying the second part of Lemma 2.3.7 to replace $\varinjlim \mathcal{I}(\mathcal{X})_{R_i}$ by $I_{\varinjlim \mathcal{X}_{R_i}}$, the essential surjectivity of the map to the 2-fiber product implies that $\text{Aut}(\mu) \rightarrow \text{Aut}(\eta)$ is surjective, while the full faithfulness implies injectivity. \square

Remark 2.3.9. If we had imposed the stronger condition that the diagonal of \mathcal{X} is locally of finite presentation, we would have that the map $\varinjlim \mathcal{X}_{R_i} \rightarrow \mathcal{X}_{\varinjlim R_i}$ is fully faithful. This is not clearly true under our weaker hypothesis.

Proposition 2.3.10. *For morphisms of quasi-algebraic stacks, the property of being locally of finite presentation is isonatural. Specifically, a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of quasi-algebraic stacks is locally of finite presentation if and only if the induced morphism $F_{\mathcal{X}} \rightarrow F_{\mathcal{Y}}$ is locally of finite presentation.*

Proof. First suppose that $F_{\mathcal{X}} \rightarrow F_{\mathcal{Y}}$ is locally of finite presentation. By Proposition 2.3.8, $\mathcal{X} \rightarrow F_{\mathcal{X}}$ and $\mathcal{Y} \rightarrow F_{\mathcal{Y}}$ are both locally of finite presentation, so applying Corollary 2.3.3 twice we see that $\mathcal{X} \rightarrow F_{\mathcal{Y}}$ and thus $\mathcal{X} \rightarrow \mathcal{Y}$ are locally of finite presentation.

Conversely, suppose that f is locally of finite presentation. We thus have that

$$\begin{array}{ccc} \varinjlim \mathcal{X}_{R_i} & \longrightarrow & \mathcal{X}_{\varinjlim R_i} \\ \downarrow & & \downarrow \\ \varinjlim \mathcal{Y}_{R_i} & \longrightarrow & \mathcal{Y}_{\varinjlim R_i} \end{array}$$

is 2-Cartesian and we wish to see that

$$\begin{array}{ccc} \varinjlim F_{\mathcal{X}}(R_i) & \longrightarrow & F_{\mathcal{X}}(\varinjlim R_i) \\ \downarrow & & \downarrow \\ \varinjlim F_{\mathcal{Y}}(R_i) & \longrightarrow & F_{\mathcal{Y}}(\varinjlim R_i) \end{array}$$

is (2-)Cartesian. Moreover, by Lemma 2.3.7, the latter diagram is (2-)Cartesian if and only if

$$\begin{array}{ccc} F_{\varinjlim \mathcal{X}_{R_i}} & \longrightarrow & F_{\mathcal{X}}(\varinjlim R_i) \\ \downarrow & & \downarrow \\ F_{\varinjlim \mathcal{Y}_{R_i}} & \longrightarrow & F_{\mathcal{Y}}(\varinjlim R_i) \end{array}$$

is (2-)Cartesian; that is, if and only if the first diagram remains (2-)Cartesian after passing to isomorphism classes. Although this is not true for arbitrary groupoids, it will be true under the quasi-algebraicity hypothesis. The only obstruction that could arise would be automorphisms of objects in $\mathcal{Y}_{\varinjlim R_i}$ which do not lift to automorphisms of either $\varinjlim \mathcal{Y}_{R_i}$ or $\mathcal{X}_{\varinjlim R_i}$. But we see that the hypothesis that the inertia stack of \mathcal{Y} is locally of finite presentation over \mathcal{Y} tells us precisely that every

automorphism of an object over $\mathcal{Y}_{\varinjlim R_i}$ lifts to an automorphism over $\varinjlim \mathcal{Y}_{R_i}$, giving us the desired result. \square

2.4. Formal criteria. We next show that under the mild deformation-theoretic hypotheses of quasi-algebraic stacks, the formal criteria for smoothness, unramifiedness, and étaleness can be rephrased functorially. It then follows that we can test for smooth-local properties of morphisms and of Artin stacks. We begin with a general remark on the sort of 2-commutative diagrams arising in formal and valuative criteria.

Remark 2.4.1. For a stack \mathcal{Y} and a scheme T' , a morphism $T' \rightarrow \mathcal{Y}$ is equivalent to an object $\eta \in \mathcal{Y}_{T'}$ together with a choice of pullback $i^*(\eta) \in \mathcal{Y}_T$ for all scheme morphisms $i : T \rightarrow T'$. Fix a morphism of stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$, and a morphism $i : T \rightarrow T'$ of schemes. A 2-commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{X} \\ \downarrow i & & \downarrow f \\ T' & \longrightarrow & \mathcal{Y} \end{array}$$

yields objects $\mu \in \mathcal{X}_T$ and $\eta \in \mathcal{Y}_{T'}$, a choice of pullback $i^*\eta \in \mathcal{Y}_T$, and an isomorphism $\gamma : i^*\eta \xrightarrow{\sim} f(\mu)$. Conversely, given (μ, η, γ) , and choices of arbitrary pullbacks for μ and for η yielding morphisms $T \rightarrow \mathcal{X}$ and $T' \rightarrow \mathcal{Y}$, we find that γ is precisely the data of a 2-isomorphism determining a 2-commutative diagram.

Next, a morphism $j : T' \rightarrow \mathcal{X}$ yields an object $\mu' \in \mathcal{X}_{T'}$ and a choice of pullback $i^*\mu' \in \mathcal{X}_T$, and gives rise to a 2-commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{X} \\ \downarrow i & \nearrow j & \downarrow f \\ T' & \longrightarrow & \mathcal{Y} \end{array}$$

if and only if there exist isomorphisms $\alpha : i^*\mu' \xrightarrow{\sim} \mu$ and $\beta : \eta \rightarrow f(\mu')$ such that $\gamma = i^*(\beta) \circ f(\alpha)$. Note here that the last condition makes sense because j also induces a map $T' \xrightarrow{f \circ j} \mathcal{Y}$, so we obtain also a choice of $i^*f(\mu')$, and see that we have $i^*f(\mu') = f(i^*\mu')$.

We therefore see that with f and i fixed, the statement that for every 2-commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{X} \\ \downarrow i & & \downarrow f \\ T' & \longrightarrow & \mathcal{Y} \end{array}$$

there exists a morphism $T' \rightarrow \mathcal{X}$ and isomorphisms making the new diagram 2-commutative is equivalent to the statement that all triples (μ, η, γ) as above, there exist μ', α, β as above with $\gamma = i^*(\beta) \circ f(\alpha)$. In addition, if \mathcal{X} has no non-trivial automorphisms, we see similarly that the uniqueness of a morphism $T' \rightarrow \mathcal{X}$ and isomorphisms making the new diagram 2-commutative is equivalent to uniqueness of the μ' such that there exists α, β with $\gamma = i^*(\beta) \circ f(\alpha)$.

Definition 2.4.2. A morphism $\mathcal{X} \rightarrow \mathcal{Y}$ is *formally smooth* (resp. *formally étale*, *formally unramified*) if for every nilpotent closed immersion $U_0 \hookrightarrow U$, with U the spectrum of a strictly Henselian local ring, every 2-commutative diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{Y} \end{array}$$

extends to a (resp. to exactly one, resp. to at most one) 2-commutative diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow \\ U & \longrightarrow & \mathcal{Y} \end{array}$$

The requirement that the diagrams be 2-commutative makes it transparent that these conditions are compatible with base change in \mathcal{Y} . Thus, if $\mathcal{X} \rightarrow \mathcal{Y}$ is representable, this gives a variant of the usual formal criterion for morphisms of algebraic spaces, with the only difference being the restriction to the strictly Henselian local case. As in the proof of Proposition 4.15(ii) of [17], when $\mathcal{X} \rightarrow \mathcal{Y}$ is locally of finite presentation, this variant suffices to establish that $\mathcal{X} \rightarrow \mathcal{Y}$ is smooth (resp. étale, resp. unramified). (We remind the reader that e.g., smoothness of a morphism f is by definition equivalent to the formal criterion of smoothness for f combined with the local finite presentation of f , as in Definition 17.3.1 of [10]. This holds true for algebraic spaces and Artin stacks; the usual local descriptions of such morphisms then follow from the compatibility of the conditions with various kinds of diagrams and the classical results for schemes.)

Proposition 2.4.3. *Let X be an affine scheme, and \mathcal{Y} a quasi-algebraic stack, and suppose we are given a morphism $f : X \rightarrow \mathcal{Y}$, locally of finite presentation. Then f is smooth (respectively, unramified, étale) if and only if for every nilpotent closed immersion $U_0 \rightarrow U$ of strictly Henselian local affine schemes, every morphism $U_0 \rightarrow X$, and every object $\eta \in F_{\mathcal{Y}}(U \amalg_{U_0} X)$ pulling back to $f \in F_{\mathcal{Y}}(X)$, there exists (respectively, there is at most one, there exists a unique) $\mu : U \rightarrow X$ such that $(\text{id} \amalg \mu)^*\eta = \delta^*p^{1*}\eta$ in the diagram*

$$\begin{array}{ccc} F_{\mathcal{Y}}(U \amalg_{U_0} X) & & \\ p^{1*} \downarrow & \searrow (\text{id} \amalg \mu)^* & \\ F_{\mathcal{Y}}(U) & \xrightarrow{\delta^*} & F_{\mathcal{Y}}(U \amalg_{U_0} U). \end{array}$$

Here $\delta : U \amalg_{U_0} U \rightarrow U$ is the codiagonal, and $p^1 : U \rightarrow U \amalg_{U_0} X$ is the first inclusion.

Notice that because X is a scheme and \mathcal{Y} is quasi-algebraic, f is necessarily representable, so smooth, unramified and étale are well-defined properties.

Proof. The key assertions are that morphisms $U_0 \rightarrow U$ and $U_0 \rightarrow X$, together with objects η as above, are equivalent in the sense of Remark 2.4.1 to 2-commutative diagrams

$$\begin{array}{ccc} U_0 & \longrightarrow & X \\ \downarrow & & \downarrow f \\ U & \longrightarrow & \mathcal{Y} \end{array}$$

as in the formal criteria of Definition 2.4.2, and that a map $\mu : U \rightarrow X$ makes a 2-commutative diagram if and only if $(\text{id} \amalg \mu)^*\eta = \delta^*p^{1*}\eta$. Given these assertions, the proposition follows immediately from the standard formal criteria in the context of representable morphisms of stacks.

Checking these assertions relies on the fact that because of the quasi-algebraicity hypothesis, $\mathcal{Y}_{U \amalg_{U_0} X}$ is equivalent to $\mathcal{Y}_U \times_{\mathcal{Y}_{U_0}} \mathcal{Y}_X$, so that $F_{\mathcal{Y}}(U \amalg_{U_0} X)$ is described by triples (ζ, f, φ) where $\zeta \in \mathcal{Y}_U$, $f \in \mathcal{Y}_X$, and $\varphi : \zeta|_{U_0} \xrightarrow{\sim} f|_{U_0}$. Two such triples are equivalent if there are isomorphisms of ζ and of f commuting with φ (abusing notation slightly, we henceforth consider f to be an object of \mathcal{Y}_X). Since we have fixed f in advance, we see that our $\eta \in F_{\mathcal{Y}}(U \amalg_{U_0} X)$ is equivalent to $\zeta \in \mathcal{Y}_U$ together with $\varphi : \zeta|_{U_0} \xrightarrow{\sim} f|_{U_0}$, which together with the maps $U_0 \rightarrow U$ and $U_0 \rightarrow X$ corresponds to the data of a 2-commutative diagram as above, as asserted.

Similarly, $F_{\mathcal{Y}}(U \amalg_{U_0} U)$ consists of pairs of objects $\zeta, \zeta' \in \mathcal{Y}_U$ together with an isomorphism $\varphi' : \zeta|_{U_0} \xrightarrow{\sim} \zeta'|_{U_0}$, once again up to pairs of isomorphisms commuting with φ' . Writing $\eta = (\zeta, f, \varphi)$ as above, we have that $(\text{id} \amalg \mu)^*\eta$ consists of the pair ζ, μ^*f glued via φ and the canonical isomorphism

$(\mu^* f)|_{U_0} \xrightarrow{\sim} f|_{U_0}$. On the other hand, $\delta^* p^{1*} \eta$ consists of ζ glued to itself along the identity. For these to be isomorphic means we have isomorphisms $\alpha_1 : \zeta \xrightarrow{\sim} \zeta$ and $\alpha_2 : \zeta \xrightarrow{\sim} \mu^* f$ such that $\varphi \circ \alpha_1|_{U_0} = \alpha_2|_{U_0} \circ \text{id} = \alpha_2|_{U_0}$, or equivalently, such that $\varphi = (\alpha_2 \circ \alpha_1^{-1})|_{U_0}$, and we see that this is precisely equivalent to the 2-commutativity of the diagram produced by adding μ . \square

In order to use Proposition 2.4.3 to detect the presence of a smooth cover, we also need the following straightforward result about the isonaturality of surjectivity.

Lemma 2.4.4. *A representable morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks is surjective if and only if the associated map of functors is surjective on geometric points.*

We can now prove all our desired results on representable morphisms.

Proof of Theorem 2.1.1. The assertions on locally of finite presentation and surjectivity are Lemma 2.3.5 and Lemma 2.4.4, respectively. Suppose \mathcal{X} is quasi-algebraic and U is a scheme. We need to show that smoothness, unramifiedness, and étaleness of a morphism $U \rightarrow \mathcal{X}$ are all isonatural. However, $U \rightarrow \mathcal{X}$ locally of finite presentation is smooth (respectively, unramified, étale) if and only if there exists an affine cover $\{U_i\}$ of U such that each $U_i \rightarrow \mathcal{X}$ is smooth (respectively, unramified, étale), and these properties are isonatural for $U_i \rightarrow \mathcal{X}$ by Proposition 2.4.3. \square

We can now conclude the following.

Corollary 2.4.5. *Given a quasi-algebraic stack \mathcal{X} , the existence of a scheme U and a smooth (respectively, étale) surjection $U \rightarrow \mathcal{X}$ is isonatural.*

In particular, the property of being an Artin or Deligne-Mumford stack is isonatural.

Moreover, the smoothness of a morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks is isonatural.

Proof. The first assertion follows immediately from Theorem 2.1.1. We then conclude that being Artin is isonatural because a quasi-algebraic stack is an Artin stack if and only if it has a smooth cover by a scheme. Similarly, being Deligne-Mumford is isonatural because a quasi-algebraic stack is Deligne-Mumford if and only if it has an étale cover by a scheme; see Definition 4.1 of [17], noting that the algebraic space X of *loc. cit.* can be replaced by a scheme by Definition 1.1 of [17].

Next, suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of Artin stacks. We note that f is smooth if and only if there exists a smooth cover $U \rightarrow \mathcal{X}$ by a scheme U such that $U \rightarrow \mathcal{Y}$ is smooth, but we just saw that both of these properties are isonatural, showing that smoothness of f is isonatural. \square

This immediately allows us to finish proving isonaturality of nearly all properties of quasi-algebraic stacks.

Corollary 2.4.6. *If P is a property of Artin stacks which is local for the smooth topology then P is isonatural. Furthermore, quasi-compactness of Artin stacks is isonatural.*

Proof. The first assertion is trivial from Corollary 2.4.5. Next, although quasi-compactness is not smooth local, it is defined (Definition 4.7.2 of [17]) in terms of the existence of a smooth cover by a quasi-compact scheme, so it is likewise isonatural. \square

It thus follows that every property of Artin stacks listed on p. 31 of [17] is isonatural. We can also use isonaturality of smooth covers to recognize a range of properties of morphisms of Artin stacks.

Lemma 2.4.7. *Let P be a property of morphisms of schemes such that smooth covers have P , and in fact P is local in the smooth topology (as in p. 33 of [17]), and further assume that P is stable under composition and base extension.*

Then a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks has P if and only if there exist smooth covers $T' \rightarrow \mathcal{X}$, $T \rightarrow \mathcal{Y}$ such that $T' \rightarrow \mathcal{Y}$ factors through T , with the map $T' \rightarrow T$ having P .

Proof. If f has P , we let $T \rightarrow \mathcal{Y}$ be any smooth cover, and T' any smooth cover of the fiber product $T \times_{\mathcal{Y}} \mathcal{X}$, and by definition (Definition 4.14 of [17]), the map $T' \rightarrow T$ will have P .

Conversely, suppose the covers T, T' exist. Note that $T' \rightarrow \mathcal{Y}$ is then a composition of morphisms having P , so has P . We then check that $T \times_{\mathcal{Y}} T'$ is a smooth cover of $T' \times_{\mathcal{Y}} \mathcal{X}$, and the natural map $T \times_{\mathcal{Y}} T' \rightarrow T$ has P , so by definition, we conclude that f has P . \square

We immediately conclude the following from the lemma and Corollary 2.4.5.

Corollary 2.4.8. *Any property P of a morphism of schemes satisfying the hypotheses of Lemma 2.4.7 is isonatural as a property of morphisms of Artin stacks. In particular, flat, surjective, and locally of finite type are each isonatural for morphisms of Artin stacks.*

Indeed, it follows that every property of morphisms listed on p. 33 of [17] is isonatural.

Remark 2.4.9. The difference between the stack-theoretic and functor-theoretic formal criteria is easiest to see in the context of the criterion for unramifiedness. Here, neither version implies the other. For instance, if we work over an algebraically closed field k , the map $\mathrm{Spec} k \rightarrow \mathrm{BG}_m$ is ramified, but appears unramified on the level of functors, as there is no non-trivial G_m -torsor over any local scheme. On the other hand, if we take the natural quotient map $\mathbf{A}_k^1 \rightarrow [\mathbf{A}_k^1/\mu_n]$ for n prime to the characteristic of k , we have an unramified map of stacks which appears to be ramified at the origin on the level of functors, since n tangent vectors all map to the same isomorphism class of $[\mathbf{A}_k^1/\mu_n]_{k[\epsilon]/\epsilon^2}$.

In contrast, it is easy to check that the stack-theoretic formal criterion for smoothness implies the functor-theoretic version. On the other hand, one can also check that for maps of the form $\mathrm{Spec} k \rightarrow \mathrm{BG}$, the functor-theoretic formal criterion for smoothness does imply the stack-theoretic version. It is not clear how generally this equivalence might hold.

2.5. Quasi-compactness. The goal of this section is to prove the following.

Proposition 2.5.1. *Given a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks, the property that f is quasi-compact is isonatural.*

We begin with three general lemmas.

Lemma 2.5.2. *Given a stack \mathcal{Y} , the functor F induces a bijection between open substacks $\mathcal{U} \subset \mathcal{Y}$ and open subfunctors $F_{\mathcal{U}} \subset F_{\mathcal{Y}}$.*

Proof. Given an open substack, the associated map of functors gives an open subfunctor, by definition. On the other hand, given an open subfunctor F' of $F_{\mathcal{Y}}$, the fiber product $F' \times_{F_{\mathcal{Y}}} \mathcal{Y}$ is an open substack of \mathcal{Y} . This gives the desired correspondence. \square

Lemma 2.5.3. *Every Artin stack \mathcal{Y} has a cover by open, quasi-compact substacks.*

Proof. Let $Y \rightarrow \mathcal{Y}$ be a smooth cover of \mathcal{Y} by a scheme, and let $\{U_i\}$ be an affine cover of Y . The images $\{\mathcal{U}_i\}$ of the $\{U_i\}$ in \mathcal{Y} are open and quasi-compact: indeed, for a representable morphism of stacks, one can define an image subfunctor in terms of T -valued points, which for a smooth morphism will be an open subfunctor; by Lemma 2.5.2 we obtain open substacks \mathcal{U}_i with smooth covers by the U_i , which then implies also that they are quasi-compact (see Definition 4.7.2 of [17]). \square

Recall that the definition of quasi-compactness (see Definition 4.16 of [17]) is that for every $Y \rightarrow \mathcal{Y}$ with Y an affine scheme over S , the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y$ is a quasi-compact stack.

Lemma 2.5.4. *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a quasi-compact morphism of Artin stacks, and \mathcal{Y} is quasi-compact, then \mathcal{X} is also quasi-compact.*

Proof. Let $Y \rightarrow \mathcal{Y}$ be a smooth cover by a quasi-compact scheme, and $\{Y_i\}$ a finite open affine cover of Y . By the definition of quasi-compact morphism, $Y_i \times_{\mathcal{Y}} \mathcal{X}$ has a smooth cover by a quasi-compact scheme X_i . The disjoint union of the X_i then gives a quasi-compact smooth cover of \mathcal{X} . \square

Lemma 2.5.5. *A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks is quasi-compact if and only if for every open quasi-compact substack \mathcal{Y}' of \mathcal{Y} , the fiber product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is quasi-compact.*

Proof. First suppose that f is quasi-compact, and we are given \mathcal{Y}' a quasi-compact open substack of \mathcal{Y} . Then by Lemma 2.5.3, \mathcal{Y}' has a smooth cover $Y \rightarrow \mathcal{Y}'$ by a quasi-compact scheme, which we can assume without loss of generality to be affine. Since f is quasi-compact, we have that $\mathcal{X} \times_{\mathcal{Y}} Y$ is quasi-compact, and is a smooth cover of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$, so we conclude that the latter is quasi-compact.

Conversely, suppose our condition is satisfied, and suppose we have $Y \rightarrow \mathcal{Y}$, with Y an affine scheme over S . Taking a cover of \mathcal{Y} by open quasi-compact substacks \mathcal{Y}_i , the preimages Y_i of \mathcal{Y}_i in Y

form an open cover. For each i , by hypothesis we have $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'_i$ quasi-compact, so it follows by Remark 4.17(1) of [17] that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'_i \rightarrow \mathcal{Y}'_i$ is quasi-compact. Taking the base change to Y , we conclude that $\mathcal{X} \times_{\mathcal{Y}} Y_i \rightarrow Y_i$ is quasi-compact for each i , and therefore that $\mathcal{X} \times_{\mathcal{Y}} Y \rightarrow Y$ is quasi-compact. Thus $\mathcal{X} \times_{\mathcal{Y}} Y$ is quasi-compact, as desired. \square

Our last lemma is that associated functors commute with fiber products when one morphism is an open immersion.

Lemma 2.5.6. *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of stacks, and $i : \mathcal{Y}' \rightarrow \mathcal{Y}$ an open immersion, then the natural map*

$$F_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'} \rightarrow F_{\mathcal{X}} \times_{F_{\mathcal{Y}}} F_{\mathcal{Y}'}$$

is an isomorphism of functors.

Proof. Indeed, a T -object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ consists of T -objects $\eta_{\mathcal{X}}$ and $\eta_{\mathcal{Y}'}$ of \mathcal{X} and \mathcal{Y}' , together with an isomorphism $\eta_{\mathcal{X}}|_{\mathcal{Y}} \xrightarrow{\sim} \eta_{\mathcal{Y}'}|_{\mathcal{Y}}$. In general, one could have two such objects glued by two different isomorphisms which are not related by automorphisms of \mathcal{X} and \mathcal{Y}' . However, when \mathcal{Y}' is an open substack of \mathcal{Y} , the natural map $\text{Aut}(\eta_{\mathcal{Y}'}) \rightarrow \text{Aut}(\eta_{\mathcal{Y}'}|_{\mathcal{Y}})$ is surjective, so this does not occur. Therefore, when we pass to isomorphism classes, we get the desired isomorphism of functors. \square

Finally, we can prove Proposition 2.5.1.

Proof of Proposition 2.5.1. By Lemma 2.5.5, $f : \mathcal{X} \rightarrow \mathcal{Y}$ is quasi-compact if and only if for all \mathcal{Y}' quasi-compact open substacks of \mathcal{Y} , we have $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ quasi-compact. We know that open substacks are in natural correspondence with open subfunctors by Lemma 2.5.2 and that we can recognize when an Artin stack is quasi-compact from its functor by Corollary 2.4.6. Finally, we can recover $F_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'}$ from $F_{\mathcal{X}}, F_{\mathcal{Y}}, F_{\mathcal{Y}'}$ from Lemma 2.5.6. We thus conclude that quasi-compactness of f is isonatural, as desired. \square

2.6. Functorial valuative criteria. We conclude our tour of properties of morphisms by addressing separatedness and properness, modifying the valuative criteria slightly to obtain criteria in terms of F_f .

Definition 2.6.1. A *doubled trait* is the non-separated scheme obtained by gluing a trait to itself along the generic point.

Given a trait Q , we let T_Q denote the doubled trait associated to Q . There is a natural morphism $\chi : T_Q \rightarrow Q$. Given a stack \mathcal{X} , we will call an element $\eta \in F_{\mathcal{X}}(T_Q)$ *constant* if it has the form $\chi^* \eta'$ for some $\eta' \in F_{\mathcal{X}}(Q)$. When Q is implicit, it will be omitted from the notation.

Lemma 2.6.2. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Artin stacks, locally of finite type, with \mathcal{Y} locally Noetherian. Then separatedness of f is isonatural. Specifically, f is separated if and only if for every doubled trait T , an object of $F_{\mathcal{X}}(T)$ is constant if and only if its image in $F_{\mathcal{Y}}(T)$ is constant.*

Proof. We show that this is equivalent to the valuative criterion for separatedness (Proposition 7.8 of [17]). Let T_1 and T_2 denote the two traits (canonically identified with Q) glued to obtain T . Given a stack \mathcal{X} , consider the natural map $F_{\mathcal{X}}(T) \rightarrow F_{\mathcal{X}}(T_1) \times_{F_{\mathcal{X}}(U)} F_{\mathcal{X}}(T_2)$. Given objects $\alpha_i \in \mathcal{X}_{T_i}$, $i = 1, 2$, with isomorphism classes $\overline{\alpha_i}$, there is a bijection between the fiber of $F_{\mathcal{X}}(T)$ over $(\overline{\alpha_1}, \overline{\alpha_2})$ and the double coset space $\text{Aut}(\alpha_2) \backslash \text{Isom}(\alpha_1|_U, \alpha_2|_U) / \text{Aut}(\alpha_1)$. (We can identify $\text{Aut}(\alpha_i)$ with a subgroup of $\text{Aut}(\alpha_i|_U)$ because the diagonal of an Artin stack is assumed separated by definition.) There is a distinguished double pseudo-coset $*$ given by the subset $\text{Isom}(\alpha_1, \alpha_2)$. (By pseudo-coset we mean that $*$ is either a single double coset or is empty.) This pseudo-coset corresponds precisely to the constant objects, and is therefore functorial in $F_{\mathcal{X}}$.

With this notation, the criterion of the lemma states that for any pair of objects $\beta_i \in \mathcal{X}_{T_i}$, $i = 1, 2$, with images $\alpha_i \in \mathcal{Y}_{T_i}$, the map

$$(2) \quad \text{Aut}(\beta_2) \backslash \text{Isom}(\beta_1|_U, \beta_2|_U) / \text{Aut}(\beta_1) \rightarrow \text{Aut}(\alpha_2) \backslash \text{Isom}(\alpha_1|_U, \alpha_2|_U) / \text{Aut}(\alpha_1)$$

has the property that the full preimage of $*$ is $*$. In particular, if an isomorphism $\phi : \beta_1|_U \xrightarrow{\sim} \beta_2|_U$ has image $f(\phi)$ which extends to an isomorphism $\psi : \alpha_1 \rightarrow \alpha_2$, we see that ϕ must extend to an isomorphism

$\bar{\phi} : \beta_1 \rightarrow \beta_2$. It then follows from the separatedness of the diagonals of \mathcal{X} and \mathcal{Y} that $\bar{\phi}$ maps to ψ under f . This is precisely the valuative criterion given in Proposition 7.8 of [17].

Conversely, suppose f is separated. The valuative criterion [loc. cit.] can be stated as follows: given a trait Q with generic point U and two objects β_1 and β_2 of \mathcal{X}_Q with images α_1 and α_2 in \mathcal{Y}_Q , any isomorphism $\phi : \beta_1|_U \xrightarrow{\sim} \beta_2|_U$ whose image in \mathcal{Y}_U extends to an isomorphism $\alpha_1 \rightarrow \alpha_2$ must extend to an isomorphism $\beta_1 \xrightarrow{\sim} \beta_2$. But this property only depends upon the image of ϕ (resp. $f(\phi)$) in the double coset space $\text{Aut}(\beta_2) \backslash \text{Isom}(\beta_1|_U, \beta_2|_U) / \text{Aut}(\beta_1)$ (resp. $\text{Aut}(\alpha_2) \backslash \text{Isom}(\alpha_1|_U, \alpha_2|_U) / \text{Aut}(\alpha_1)$). Thus, we find that the preimage of $*$ under the natural map (2) is $*$, as desired. \square

We next move on to properness.

Lemma 2.6.3. *Suppose $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of Artin stacks. Given a trait T with generic point U , and a 2-commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow f \\ T & \longrightarrow & \mathcal{Y}, \end{array}$$

consider the induced commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & F_{\mathcal{X}} \\ \downarrow & \nearrow & \downarrow F_f \\ T & \longrightarrow & F_{\mathcal{Y}}. \end{array}$$

Then:

- (1) *every square of the second form is induced by one of the first form;*
- (2) *if the first square admits a morphism $T \rightarrow \mathcal{X}$ making the entire diagram 2-commutative, then the induced map $T \rightarrow F_{\mathcal{X}}$ gives a commutative diagram when added to the second square;*
- (3) *if \mathcal{Y} has proper inertia, then we have conversely that any morphism $T \rightarrow F_{\mathcal{X}}$ making the second diagram commutative yields a morphism $T \rightarrow \mathcal{X}$ making the first diagram 2-commutative.*

Proof. The first assertion is trivial. For the remaining claims, the key issue to consider is that, as discussed in Remark 2.4.1, the 2-commutative square above consists of $\mu_0 \in \mathcal{X}_U$, $\eta \in \mathcal{Y}_T$, and an isomorphism $\alpha : f(\mu_0) \rightarrow \eta|_U$, and a map $T \rightarrow \mathcal{X}$, given by $\mu \in \mathcal{X}_T$, allows the diagram to be filled in to a 2-commutative diagram if there exist isomorphisms $\beta : \mu_0 \rightarrow \mu|_U$ and $\gamma : f(\mu) \rightarrow \eta$ such that $\alpha = \gamma|_U \circ f(\beta)$. Filling in the second diagram is the same, except without the final compatibility condition on the isomorphisms. Thus, it is clear that if the first diagram may be filled in to be 2-commutative, the second one may be filled in to be commutative. Conversely, if the second one may be filled in to be commutative, $\gamma|_U \circ f(\beta) \circ \alpha^{-1} \in \text{Aut}(\eta|_U)$, and if $\text{Aut}(\eta) \rightarrow \text{Aut}(\eta|_U)$ is surjective, we can modify γ to obtain $\alpha = \gamma|_U \circ f(\beta)$, giving the desired 2-commutativity. \square

An almost immediate consequence of the two lemmas is the following.

Corollary 2.6.4. *Properness is isonatural for morphisms $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks, where \mathcal{Y} is further supposed to be locally Noetherian and to have proper inertia.*

Specifically, f is proper if and only if it is locally of finite type, quasi-compact, and separated, and if for every trait T with generic point U , with morphisms $T \rightarrow F_{\mathcal{Y}}$ and $U \rightarrow F_{\mathcal{X}}$, there exists a trait T' with generic point U' , obtained by normalizing T' inside the finite field extension given by $U' \rightarrow U$, and morphisms making the following diagram commute:

$$\begin{array}{ccccc} U' & \longrightarrow & U & \longrightarrow & F_{\mathcal{X}} \\ \downarrow & \nearrow & \downarrow & & \downarrow F_f \\ T' & \longrightarrow & T & \longrightarrow & F_{\mathcal{Y}}. \end{array}$$

Proof. We first remark that there is a standard stack version of the valuative criterion for properness. This is stated as (iii) of Theorem 7.10 of [17], using also Proposition 7.12 of *loc. cit.*, and noting that condition (*) of *ibid.* is always satisfied, thanks to the main theorem of [22].

Because being separated, quasi-compact, or locally of finite type are all isonatural, we need only check that our asserted valuative criterion is equivalent to the usual valuative criterion cited above. But that follows immediately from the previous lemma. \square

Note that in particular, if \mathcal{Y} is locally Noetherian scheme or algebraic space, properness is isonatural. We also use doubled traits to see that having proper inertia is isonatural, which completes our list of isonatural properties of stacks.

Proof of Corollary 2.1.3. That being an Artin or Deligne-Mumford stack is isonatural is part of Corollary 2.4.5. Next, because a stack is a gerbe (over an algebraic space) if and only if the sheafification of the associated functor is an algebraic space (see Remark 3.16(1) [17]), we also see that the property of being a gerbe is isonatural.

Corollary 2.4.6 implies that locally Noetherian, normal, reduced, regular, and quasi-compact are each isonatural for Artin stacks.

Finally, we see that having proper inertia is likewise isonatural, because if \mathcal{X} be a quasi-algebraic stack, T_1 a trait with generic point U , and $\eta \in \mathcal{X}_{T_1}$, if we let T be the doubled trait obtained by gluing T_1 to itself along U , it is easy to see that $\text{Aut}(\eta) \rightarrow \text{Aut}(\eta|_U)$ is surjective if and only if there is a unique element of $F_{\mathcal{X}}(T)$ pulling back to η under both restriction maps. \square

By using doubled traits in the criterion for universal closedness, we can further expand the range of cases in which we can treat properness, as follows.

Proposition 2.6.5. *Properness is isonatural for morphisms $f : \mathcal{X} \rightarrow \mathcal{Y}$ of Artin stacks, with \mathcal{Y} Noetherian and abelian (see Definition 1.1.1).*

Specifically, f is proper if and only if it satisfies all the conditions of Corollary 2.6.4, and if in addition, for every doubled trait T with generic point U , obtained by gluing together traits $T_1 = T_2$ along U , and given morphisms $T \rightarrow F_{\mathcal{Y}}$ and $T_1 \rightarrow F_{\mathcal{X}}$, there exists a doubled trait T' with generic point U' , obtained from $T'_1 = T'_2$ the normalization of $T_1 = T_2$ inside the finite field extension given by $U' \rightarrow U$, and morphisms making the following diagram commute:

$$\begin{array}{ccccc} T'_1 & \longrightarrow & T_1 & \longrightarrow & F_{\mathcal{X}} \\ \downarrow & \nearrow & \downarrow & & \downarrow F_f \\ T' & \longrightarrow & T & \longrightarrow & F_{\mathcal{Y}}. \end{array}$$

Proof. As before, it suffices to see that our criterion in terms of functors and doubled traits is equivalent to the usual criterion in terms of stacks, under the hypothesis that \mathcal{Y} has abelian stabilizers. Once T' is given, we can ignore the original square, and consider instead the square

$$\begin{array}{ccc} T'_1 & \longrightarrow & F_{\mathcal{X}} \\ \downarrow & \nearrow & \downarrow F_f \\ T' & \longrightarrow & F_{\mathcal{Y}}. \end{array}$$

To simplify notation and avoid the uncontrolled proliferation of $'$, when we say “after extension” $U' \rightarrow U$ we will assume we have replaced U by U' , T by T' , objects and morphisms by their appropriate pullbacks, and so forth. We will use Remark 2.4.1 to translate between the 2-commutative diagrams of the formal criterion and objects and isomorphisms of the stacks themselves.

The map $T_1 \rightarrow F_{\mathcal{X}}$ is equivalent to an object $\mu_1 \in \mathcal{X}_U$, up to isomorphism. The map $T \rightarrow F_{\mathcal{Y}}$ is equivalent to a pair of objects $\eta_1 \in \mathcal{Y}_{T_1}, \eta_2 \in \mathcal{Y}_{T_2}$, and a choice of isomorphism $\varphi : \eta_1|_U \xrightarrow{\sim} \eta_2|_U$, up to simultaneous isomorphism commuting with φ . We also assume that $f(\mu_1) \cong \eta_1$. The desired map $T \rightarrow F_{\mathcal{X}}$ is then given by extending μ_1 to a triple (μ_1, μ_2, ϕ) for $\mu \in \mathcal{X}_{T_2}$, $\phi : \mu_1|_U \xrightarrow{\sim} \mu_2|_U$, with the

additional restriction that there exist isomorphisms $\alpha_1 : f(\mu_1) \rightarrow \eta_1$ and $\alpha_2 : f(\mu_2) \rightarrow \eta_2$, satisfying $\varphi \circ \alpha_1|_U = \alpha_2|_U \circ f(\phi)$.

Suppose f is proper. We have by the earlier lemma that the conditions of Corollary 2.6.4 are satisfied, so we wish to show that our condition on doubled traits is also satisfied. Starting with $(\eta_1, \eta_2, \varphi)$ in \mathcal{B}_T and $\mu_1 \in \mathcal{X}_{T_1}$, and fixing further any $\alpha_1 : f(\mu_1) \rightarrow \eta_1$, applying the valuative criterion of properness to $\eta_2, \mu_1|_U$, and $\varphi \circ \alpha_1|_U$, after an appropriate extension there exists $\mu_2 \in \mathcal{X}_{T_2}$, $\beta_2 : \mu_1|_U \rightarrow \mu_2|_U$, $\gamma_2 : f(\mu_2) \rightarrow \eta_2$, such that $\varphi \circ \alpha_1|_U = \gamma_2|_U \circ f(\beta_2)$. Setting the above $\alpha_2 = \gamma_2$ and $\phi = \beta_2$ gives us precisely what we wanted. Note that this direction did not use any hypotheses on the stabilizer being abelian.

Conversely, suppose that f satisfies our criterion, and \mathcal{B} has abelian stabilizer groups. We then want to show that f satisfies the valuative criterion for universal closedness, and is therefore proper. Here, we are simply given $\mu_0 \in \mathcal{X}_U$, $\eta \in \mathcal{B}_{T_1}$, and $\beta : f(\mu_0) \rightarrow \eta|_U$, and we wish to show that after finite extension, there exists $\mu \in \mathcal{X}_{T_1}$ and isomorphisms $\gamma : \mu_0 \rightarrow \mu|_U$ and $\alpha : f(\mu) \rightarrow \eta$ such that $\alpha|_U \circ f(\gamma) = \beta$. We first apply the criterion of Corollary 2.6.4 to find that after extension, we have $\mu_1 \in \mathcal{X}_{T_1}$ and isomorphisms $\gamma_1 : \mu_0 \rightarrow \mu_1|_U$ and $\alpha' : f(\mu_1) \rightarrow \eta$ not necessarily satisfying any compatibility condition. We set $\eta_1 = \eta_2 = \eta$, and $\varphi : \beta \circ f(\gamma_1)^{-1} \circ (\alpha'|_U)^{-1}$. Our criterion says that after an additional extension, we have $\mu_2 \in \mathcal{X}_{T_2}$, $\phi : \mu_1|_U \rightarrow \mu_2|_U$, and $\alpha_i : f(\mu_i) \rightarrow \eta_i$ satisfying $\varphi \circ \alpha_1|_U = \alpha_2|_U \circ f(\phi)$, so we have $\beta \circ f(\gamma_1)^{-1} \circ (\alpha'|_U)^{-1} \circ \alpha_1|_U = \alpha_2|_U \circ f(\phi)$. We claim that if we set $\mu = \mu_2$, and $\gamma = \phi \circ \gamma_1$, and $\alpha = \alpha' \circ \alpha_1^{-1} \circ \alpha_2$, we obtain $\alpha|_U \circ f(\gamma) = \beta$, as desired. The key observation is that

$$\alpha = \alpha_2 \circ (\alpha_2^{-1} \circ \alpha_1) \circ (\alpha_1^{-1} \circ \alpha') \circ (\alpha_1^{-1} \circ \alpha_2).$$

Because automorphism groups are abelian, conjugation of an automorphism by any two isomorphisms yields the same result, and applying this to $(\alpha_1^{-1} \circ \alpha')|_U$ we find

$$\alpha|_U = \alpha_2|_U \circ f(\phi) \circ (\alpha_1^{-1} \circ \alpha')|_U \circ f(\phi)^{-1}.$$

Substituting above we easily obtain the desired identity. \square

We have now finished proving isonaturality of all the asserted properties of Artin stacks.

Proof of Theorem 2.1.2. Isonaturality for morphisms being locally of finite presentation is Proposition 2.3.10, smoothness follows from Corollary 2.4.5, and then locally of finite type, surjective, and flat follow from Corollary 2.4.8. Isonaturality for morphisms being quasi-compact is Proposition 2.5.1, and separated and locally of finite type when the target is locally Noetherian is Lemma 2.6.2. Finally, proper morphisms when the target is locally Noetherian and has proper inertia is covered by Corollary 2.6.4, and when the target is locally Noetherian and has abelian stabilizers is Proposition 2.6.5. \square

2.7. Functorial reconstruction of automorphism groups. In this section we describe a structure that can be used to recover the presheaf of conjugacy classes in the inertia of any quasi-algebraic stack. When the stack is abelian, this permits us to reconstruct abelian automorphism sheaves.

Definition 2.7.1. The *universal binana* $N_{2,\mathbb{Z}}$ is the proper curve over $\text{Spec } \mathbb{Z}$ obtained by gluing together two copies of $\mathbb{P}_{\mathbb{Z}}^1$ to one another transversally along the 0 and 1 sections. Given any scheme T , the *binana over T* , denoted $N_{2,T}$ is $N_{2,\mathbb{Z}} \times T$. We denote by 0_T and 1_T the images of the 0 and 1 sections.

The binana over T has the two *peel* maps $P_i^2 : \mathbb{P}_T^1 \rightarrow N_{2,T}$ for $i = 1, 2$; each is a closed immersion, and the intersection of their images is precisely $0_T \cup 1_T$.

Finally, we set the following notation: $N_{2,T}^0 := N_{2,T} \setminus 0_T$, $N_{2,T}^1 := N_{2,T} \setminus 1_T$, and $N_{2,T}^{0,1} := N_{2,T} \setminus \{0_T, 1_T\}$.

We consider objects of functors over binanas which are constant on each peel; the isomorphism classes can thus be thought of (at least informally) in terms of gluing along isomorphisms over the 0 and 1 sections.

Definition 2.7.2. Let F be a functor from S -schemes to sets, T a scheme over S , and $\eta \in F(T)$. Given a T -scheme T' , we say an object $\eta' \in F(T')$ is η -trivial if $\eta' = \eta|_{T'}$.

Definition 2.7.3. Given $\eta \in F(T)$, an η -binana is an object $\tilde{\eta} \in F(N_{2,T})$ satisfying the following conditions:

- (1) $\tilde{\eta}|_{N_{2,T} \setminus 0_T}$ and $\tilde{\eta}|_{N_{2,T} \setminus 1_T}$ are both η -trivial;
- (2) $(P_1^2)^*(\tilde{\eta})$ and $(P_2^2)^*(\tilde{\eta})$ are η -trivial.

Definition 2.7.4. The functor sending $T' \rightarrow T \in T\text{-Sch}$ to the set of $\eta_{T'}$ -binanas will be called the *functor of η -binanas* and denoted $\text{Bin}(\eta)$.

Definition 2.7.5. Let G be a sheaf of groups on a site Ξ . The presheaf sending R in Ξ to the set of conjugacy classes of $G(R)$ will be called the *presheaf of conjugacy classes of G* and denoted $\text{Conj}(G)$.

Lemma 2.7.6. *If the sheaf of groups G in Definition 2.7.5 is abelian then there is a canonical isomorphism of presheaves $G \rightarrow \text{Conj}(G)$.*

Proof. This follows immediately from the definition. \square

Proposition 2.7.7. *Let \mathcal{X} be a quasi-algebraic stack and $\tilde{a} \in \mathcal{X}_T$ an object with isomorphism class $a \in F_{\mathcal{X}}(T)$.*

- (1) *there is a canonical isomorphism of functors*

$$\text{Bin}(a) \xrightarrow{\sim} \text{Conj}(\text{Aut}(\tilde{a}));$$

- (2) *if $\text{Aut}(\tilde{a})$ is an abelian sheaf there is a canonical isomorphism*

$$\text{Bin}(a) \xrightarrow{\sim} \text{Aut}(\tilde{a}).$$

Moreover, these isomorphisms are functorial in the pair (\mathcal{X}, \tilde{a}) .

Proof. The hypothesis that \mathcal{X} is quasi-algebraic implies in particular that $\text{Aut}(\tilde{a})$ is a group scheme over T .

Condition (1) in the definition of an a -binana implies that a -binanas may be understood in terms of gluing a -trivial families on $N_{2,T}^0$ and $N_{2,T}^1$ along the intersection $N_{2,T}^{0,1}$, which is isomorphic to $(\mathbb{P}_T^1 \setminus \{0_T, 1_T\}) \amalg (\mathbb{P}_T^1 \setminus \{0_T, 1_T\})$. Thus, an a -binana is determined by the data of two sections φ_1, φ_2 of $\text{Aut}(\tilde{a})$ over $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$; we think of the pair (φ_1, φ_2) as a section of $\text{Aut}(\tilde{a})$ over $N_{2,T}^{0,1}$. Condition (2) is precisely the restriction that each of these sections must be expressible as the difference of sections of $\text{Aut}(\tilde{a})$ over $\mathbb{P}_T^1 \setminus 0_T$ and over $\mathbb{P}_T^1 \setminus 1_T$.

Two a -binanas obtained from gluing along (φ_1, φ_2) and (φ'_1, φ'_2) are isomorphic if and only if there exists sections α_0, α_1 of $\text{Aut}(\tilde{a})$ over $N_{2,T}^0$ and $N_{2,T}^1$ respectively, such that $(\varphi'_1, \varphi'_2) \circ \alpha_0|_{N_{2,T}^{0,1}} = \alpha_1|_{N_{2,T}^{0,1}} \circ (\varphi_1, \varphi_2)$.

We now construct a map from $\text{Aut}(\tilde{a})$ to the set of a -binanas. Given $\varphi \in \text{Aut}(\tilde{a})$ (over the base scheme T itself), we glue along the constant automorphisms (id, φ) to obtain a binana. Being constant, there is no problem with extending either of them to \mathbb{P}_T^1 , so condition (2) is satisfied, and we obtain an a -binana. We wish to show that two binanas obtained in this way from φ and φ' are the same if and only if φ and φ' are conjugate to one another in $\text{Aut}(\tilde{a})$, and that every a -binana is obtained in this way.

For the first assertion, φ and φ' yield the same a -binana if and only if there exist α_0 and α_1 as above with $(\text{id}, \varphi) \circ \alpha_0|_{N_{2,T}^{0,1}} = \alpha_1|_{N_{2,T}^{0,1}} \circ (\text{id}, \varphi')$, which is equivalent to $\alpha_0 = \alpha_1$ after restriction to the first copy of $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$, and $\alpha_0 = \varphi^{-1}\alpha_1\varphi'$ after restriction to the second copy of $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$.

Suppose such α_i exist. Since φ and φ' are constant, this implies that α_0 and α_1 can be extended over the partial normalizations of $N_{2,T}$ over 0_T and 1_T respectively, which implies they are themselves constant, since $\text{Aut}(\tilde{a})$ is a group scheme. Hence, by looking at the first copy of $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$, the α_i are also globally equal. Looking at the second copy of $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$ gives us $\varphi = \alpha_1\varphi'\alpha_0^{-1} = \alpha_0\varphi'\alpha_0^{-1}$, and since α_0 is constant, we find that φ and φ' are conjugate, as desired. Conversely, it is clear that if $\varphi = \alpha\varphi'\alpha^{-1}$, setting α_0 and α_1 equal to the constant sections obtained from α yields an isomorphism between the a -binanas obtained from φ and φ' .

It remains to see that given an a -binana coming from a pair (φ_1, φ_2) , there is some $\varphi \in \text{Aut}(\tilde{a})$ yielding the same a -binana. By hypothesis, there exist $\alpha_{0,1}, \alpha_{0,2}$ sections of $\text{Aut}(\tilde{a})$ over $\mathbb{P}_T^1 \setminus 0_T$ and $\alpha_{1,1}, \alpha_{1,2}$ sections of $\text{Aut}(\tilde{a})$ over $\mathbb{P}_T^1 \setminus 1_T$ with $\varphi_1 = \alpha_{1,1}^{-1} \circ \alpha_{0,1}$ and $\varphi_2 = \alpha_{1,2}^{-1} \circ \alpha_{0,2}$ after restriction to $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$. If we modify $\alpha_{0,2}$ and $\alpha_{1,2}$ by the constant sections coming from $\alpha_{0,1}|_{0_T} \circ \alpha_{0,2}^{-1}|_{0_T}$, we can glue $\alpha_{0,1}$ and $\alpha_{0,2}$ to obtain a section α_0 of $\text{Aut}(\tilde{a})$ over $N_{2,T}^0$. Define α_1 over $N_{2,T}^1$ to be obtained by

gluing $\alpha_{1,1}$ to $(\alpha_{1,1}|_{1_T} \circ \alpha_{1,2}|_{1_T}^{-1}) \circ \alpha_{1,2}$. Then we see that α_0 and α_1 define an isomorphism between the a -binana obtained by gluing along (φ_1, φ_2) and the one obtained from $\alpha_{1,1}|_{1_T} \circ \alpha_{1,2}|_{1_T}^{-1} \in \text{Aut}(\tilde{a})$. \square

In order to recover the composition law on automorphism groups, we now introduce a further structure.

Definition 2.7.8. The *universal trinana* $N_{3,\mathbb{Z}}$ is the proper curve over $\text{Spec } \mathbb{Z}$ obtained by gluing together three copies of $\mathbb{P}_{\mathbb{Z}}^1$ transversally along the 0 and 1 sections. Given a scheme T , the *trinana over T* , denoted $N_{3,T}$, is $N_{3,\mathbb{Z}} \times T$. As before, we denote by 0_T and 1_T the images of the 0 and 1 sections.

The trinana over T has three peel maps $P_i^3 : \mathbb{P}_T^1 \rightarrow N_{3,T}$ for $i = 1, 2, 3$; each is again a closed immersion, and the intersection of any two of their images is precisely $0_T \cup 1_T$. Finally, there are three *bipeel* maps $P_{i,j} : N_{2,T} \rightarrow N_{3,T}$ for $(i, j) = (1, 2), (1, 3), (2, 3)$. Each is again a closed immersion, and we have $P_{i,j} \circ P_1^2 = P_i^3$ and $P_{i,j} \circ P_2^2 = P_j^3$.

Finally, we set the following notation: $N_{3,T}^0 := N_{3,T} \setminus 0_T$, $N_{3,T}^1 := N_{3,T} \setminus 1_T$, and $N_{3,T}^{0,1} := N_{3,T} \setminus \{0_T, 1_T\}$.

Definition 2.7.9. Given $\eta \in F(T)$, an η -*trinana* is an object η' of $F(N_{3,T})$ such that

- (1) $\tilde{\eta}|_{N_{3,T} \setminus 0_T}$ and $\tilde{\eta}|_{N_{3,T} \setminus 1_T}$ are both η -trivial;
- (2) $(P_i^3)^*(\eta')$ is η -trivial for $i = 1, 2, 3$.

Definition 2.7.10. Given $\eta \in F(T)$, the functor which assigns to any $T' \rightarrow T \in T\text{-Sch}$ the set of $\eta_{T'}$ -trinanas will be called the *functor of η -trinanas* and denoted $\text{Trin}(\eta)$.

Definition 2.7.11. Given $\eta \in F(T)$ as above, an η -binana $\tilde{\eta}$ and an η -binana $\tilde{\eta}'$, an $(\tilde{\eta}, \tilde{\eta}')$ -*trinana* is an object $\mu \in F(N_{3,T})$ such that:

- (1) $\mu|_{N_{3,T}^0}$ and $\mu|_{N_{3,T}^1}$ are both η -trivial;
- (2) we have $P_{1,2}^*\mu = \tilde{\eta}$ and $P_{2,3}^*\mu = \tilde{\eta}'$.

Definition 2.7.12. The three bipeel morphisms yield a diagram of functors

$$(3) \quad \begin{array}{ccc} \text{Trin}(\eta) & \xrightarrow{P_{1,2}^* \times P_{2,3}^*} & \text{Bin}(\eta) \times \text{Bin}(\eta) \\ P_{1,3}^* \downarrow & & \\ & & \text{Bin}(\eta) \end{array}$$

which we will call the *fundamental diagram of η -nanas*.

Proposition 2.7.13. Given a quasi-algebraic stack \mathcal{X} , a scheme T , and $\tilde{a} \in \mathcal{X}_T$ with image $a \in F_{\mathcal{X}}(T)$, the group $\text{Aut}(\tilde{a})$ is abelian if and only if the horizontal arrow in the fundamental diagram of Definition 2.7.12 is a bijection. In this case, the composition law is given by the vertical arrow in the fundamental diagram, via the isomorphism of Proposition 2.7.7(2).

Proof. As in the case of a -binanas, we see that condition (1) for a trinana means that it is determined by a triple of sections (ψ_1, ψ_2, ψ_3) of $\text{Aut}(\tilde{a})$ over $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$. If b and b' are a -binanas represented by (φ_1, φ_2) and (φ'_1, φ'_2) respectively, then condition (2) simply requires isomorphisms between the binanas obtained from (ψ_1, ψ_2) and (φ_1, φ_2) , and (ψ_2, ψ_3) and (φ'_1, φ'_2) . Moreover, we know from the proof of the previous proposition that without loss of generality, we can set $(\varphi_1, \varphi_2) = (\text{id}, \varphi)$ and $(\varphi'_1, \varphi'_2) = (\text{id}, \varphi')$, where φ and φ' are constant sections of $\text{Aut}(\tilde{a})$. Now, it is easy to check that if we set $(\psi_1, \psi_2, \psi_3) = (\text{id}, \varphi_1, \varphi_1\varphi_2)$ we obtain an (b, b') -trinana, so our assertion is that this is the unique possibility if and only if $\text{Aut}(\tilde{a})$ is abelian.

One direction is clear: if $\text{Aut}(\tilde{a})$ is non-abelian, then by choosing φ in a non-trivial conjugacy class, say with $\gamma^{-1}\varphi\gamma \neq \varphi$, then we see by comparing the two representations of the same a -binanas given by $\varphi' = \varphi^{-1}$ and $\varphi' = \gamma^{-1}\varphi^{-1}\gamma$, that we have the two (b, b') -trinanas given by $(\text{id}, \varphi, \text{id})$ and $(\text{id}, \varphi, \varphi\gamma^{-1}\varphi^{-1}\gamma)$, and these cannot be isomorphic because their pullbacks under $P_{1,3}$ yield non-isomorphic a -binanas.

It remains to show that if $\text{Aut}(\tilde{a})$ is abelian, then an (b, b') -trinana given by (ψ_1, ψ_2, ψ_3) is necessarily isomorphic to the one given by $(\text{id}, \varphi_1, \varphi_1\varphi_2)$. We therefore wish to construct β_0 and β_1 , sections of

$\text{Aut}(\tilde{a})$ over $N_{3,T}^0$ and $N_{3,T}^1$ respectively, such that $(\psi_1, \psi_2, \psi_3) \circ \beta_0|_{N_{3,T}^{0,1}} = \beta_1|_{N_{3,T}^{0,1}} \circ (\text{id}, \varphi_1, \varphi_1\varphi_2)$. We are given α_0 and α'_0 over $N_{2,T}^0$ and α_1 and α'_1 over $N_{2,T}^1$, such that $(\psi_1, \psi_2) \circ \alpha_0|_{N_{2,T}^{0,1}} = \alpha_1|_{N_{2,T}^{0,1}} \circ (\text{id}, \varphi_1)$ and $(\psi_2, \psi_3) \circ \alpha'_0|_{N_{2,T}^{0,1}} = \alpha'_1|_{N_{2,T}^{0,1}} \circ (\text{id}, \varphi_2)$.

We define β_0 to be α_0 on the first and second peels, and $\alpha'_0(\alpha'_0|_{0_T})^{-1}\alpha_0|_{0_T}$ on the third peel. Similarly, we set β_1 to be α_1 on the first and second peels, and $\alpha'_1(\alpha'_1|_{1_T})^{-1}\alpha_1|_{1_T}$ on the third peel. Using the abelian hypothesis, we note that on the second peel, restricted to $\mathbb{P}_T^1 \setminus \{0_T, 1_T\}$, we have $\alpha'_1\alpha_1^{-1} = \varphi_1\alpha'_0\alpha_0^{-1}$. Since φ_1 is constant, this means that $\alpha'_1\alpha_1^{-1}$ on the second peel extends over 0_T to all of \mathbb{P}_T^1 , and must therefore be constant. Thus, $\alpha'_0\alpha_0^{-1}$ is also constant, and we conclude the identity $(\alpha_1|_{1_T})^{-1}\alpha'_1|_{1_T}(\alpha'_0|_{0_T})^{-1}\alpha_0|_{0_T} = \varphi_1$. From this, it is easy to check that β_0 and β_1 define the required isomorphism of trinanases. \square

Remark 2.7.14. It is a general fact that any morphism of stacks which induces a bijection on isomorphism classes and isomorphisms on all automorphism groups is an isomorphism. It thus follows from the previous propositions that if we have a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of quasi-algebraic stacks inducing an isomorphism $F_{\mathcal{X}} \xrightarrow{\sim} F_{\mathcal{Y}}$, and if either \mathcal{X} or \mathcal{Y} is abelian, then f is an isomorphism.

Corollary 2.7.15. *If $\mathcal{X} \rightarrow X$ is an abelian quasi-algebraic gerbe then the band of \mathcal{X} can be recovered from $F_{\mathcal{X}}$.*

Proof. Write Φ for the category fibered in groupoids on X associated to the functor $F_{\mathcal{X}}$, so that there is a diagram of functors

$$\mathcal{X} \xrightarrow{c} \Phi \xrightarrow{p} X\text{-Sch}.$$

Let \mathbf{Ab} be the category of abelian groups.

Viewing the inertia stack of \mathcal{X} as a sheaf on the natural site of \mathcal{X} yields a functor $\iota : \mathcal{X}^\circ \rightarrow \mathbf{Ab}$. According to Propositions 2.7.7 and 2.7.13, there is a functor $\Gamma : \Phi^\circ \rightarrow \mathbf{Ab}$ such that ι is isomorphic to $\Gamma \circ c$. (The underlying set of Γ is just Bin .)

Moreover, since \mathcal{X} is an abelian gerbe, there is an abelian sheaf $\Lambda : X\text{-Sch}^\circ \rightarrow \mathbf{Ab}$ on X (the band of \mathcal{X}) and an isomorphism $\iota \xrightarrow{\sim} \Lambda \circ p \circ c$. We find an isomorphism $\psi : \Gamma \circ c \xrightarrow{\sim} \Lambda \circ p \circ c$. Let $\chi : U \rightarrow X$ be an fppf covering such that there is a lift $q : U \rightarrow \mathcal{X}$; let \bar{q} denote the composition $c \circ q$. Composing with ψ yields an isomorphism $\Gamma \circ c \circ q \xrightarrow{\sim} \Lambda \circ p \circ c \circ q$, which (via the natural isomorphisms) yields an isomorphism $\Gamma \circ \bar{q} \xrightarrow{\sim} \Lambda \circ \chi$.

This isomorphism tells us that (via Bin and diagram (3)) we can recover $\Lambda \circ \chi$ for some fppf covering $\chi : U \rightarrow X$. Since any abelian sheaf on X is uniquely determined by its values on the category of U -schemes (a simple consequence of the sheaf property), it follows that Λ is uniquely determined up to isomorphism by $F_{\mathcal{X}}$, as desired. \square

Corollary 2.7.16. *The associated functor of a quasi-algebraic stack \mathcal{X} is a sheaf if and only if \mathcal{X} has no non-trivial automorphisms.*

In particular, if \mathcal{X} is an Artin stack, then the associated functor is a sheaf if and only if \mathcal{X} is an algebraic space.

Proof. Certainly, if \mathcal{X} has no non-trivial automorphisms, then the stack condition implies its associated functor is a sheaf. Conversely, if \mathcal{X} is a sheaf, we see from the above argument that every automorphism group must be trivial: if $\text{Aut}(\eta) \neq \{1\}$ for some T and $\eta \in \mathcal{X}_T$, we would have at least two non-isomorphic η -binanas, which by definition become isomorphic after restriction to $N_{2,T}^0$ and $N_{2,T}^1$. This would violate the sheaf condition, so $\text{Aut}(\eta) = \{1\}$.

For the last assertion, we use that by Corollary 8.1.1 of [17], an Artin stack is an algebraic space if and only if it has no non-trivial automorphisms. \square

3. ISONATURAL STACKS

In this section, we examine several classes of stacks, showing that within these classes, stacks are uniquely determined by their associated functors. We also show that one can recognize whether a given quasi-algebraic stack lies in each class, proving that a stack lying in any of the given classes is isonatural.

3.1. Summary of results. In order to give the precise statements of our results, we make the following definitions.

Definition 3.1.1. We say that an algebraic space X is *strongly R1* if X is Noetherian, integral, separated, and R1.

Definition 3.1.2. Let $G = \bigoplus \mu_N$ be a diagonalizable group scheme. A cohomology class $\alpha \in H^2(X, G) = \bigoplus H^2(X, \mu_N)$ will be called *Brauer* if the image of each component via $H^2(X, \mu_N) \rightarrow H^2(X, \mathbf{G}_m)$ lies in $\text{Br}(X)$. A G -gerbe \mathcal{X} will be called Brauer if its cohomology class $[\mathcal{X}] \in H^2(X, G)$ is Brauer.

Remark 3.1.3. According to a theorem of Gabber [3], if the connected components of X are quasi-compact separated schemes admitting ample invertible sheaves then every class as in Definition 3.1.2 is Brauer.

Furthermore, whether or not a cohomology class is Brauer is independent of the choice of representation of G as a direct sum.

Definition 3.1.4. Given a quasi-algebraic stack \mathcal{X} , the *clean locus* of \mathcal{X} , denoted $\text{cl}(\mathcal{X})$ is the locus over which the inertia stack $\mathcal{I}(\mathcal{X}) \rightarrow \mathcal{X}$ is an isomorphism (i.e., the locus parametrizing objects with trivial automorphism sheaves)

Proposition 3.1.5. *If \mathcal{X} has proper inertia then the clean locus of \mathcal{X} is an open substack, and the inclusion map $\text{cl}(\mathcal{X}) \rightarrow \mathcal{X}$ is quasi-compact.*

Note that the inclusion of an open substack is representable by definition, so it makes sense to ask whether or not the inclusion morphism is quasi-compact, as in (3.10) of [17].

Proof. This reduces to the following: if $f : G \rightarrow X$ is a proper group scheme of finite presentation over an affine scheme then the subsheaf T of X over which f is an isomorphism is represented by a quasi-compact open immersion $U \hookrightarrow X$. The (big) sheaf T is compatible with base change (by definition). Since f is of finite presentation, we may assume that X is the spectrum of a finite-type \mathbf{Z} -algebra (and, in particular, Noetherian). The result is then given by Proposition 4.6.7(ii) of [8]. \square

Remark 3.1.6. In the absence of the hypothesis that $G \rightarrow X$ is proper, it is easy to see that the clean locus need not be open.

Definition 3.1.7. A quasi-algebraic stack \mathcal{X} is *bald* if it has proper diagonal and the clean locus $\text{cl}(\mathcal{X}) \subset \mathcal{X}$ is schematically dense.

Recall that an open substack $\mathcal{U} \hookrightarrow \mathcal{X}$ is *schematically dense* if any closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$ which contains \mathcal{U} is necessarily equal to \mathcal{X} .

The remainder of the present paper is devoted to proving the following.

Theorem 3.1.8. *The following quasi-algebraic stacks are isonatural:*

- (1) *bald Artin stacks;*
- (2) *BG, where G is a finite étale group space over a locally Noetherian algebraic space;*
- (3) *BG, where G is an abelian group space locally of finite presentation over an algebraic space;*
- (4) *Brauer G -gerbes over a strongly R1 algebraic space, with G a diagonalizable finite group scheme;*
- (5) *Brauer \mathbf{G}_m -gerbes over an algebraic space.*

Part (1) is treated in Section 3.2. Part (2) is Proposition 3.3.20, and we have already proved part (3) because we can recover abelian stabilizers functorially by Theorem 2.1.4. We prove part (4) in Proposition 3.4.11, and part (5) in Proposition 3.5.1.

Before embarking on the proof of the theorem, we note that the results of Section 2 are sufficiently fine to sift out all the classes of stacks appearing in Theorem 3.1.8 from their functors.

Proposition 3.1.9. *Each of the classes of stacks in Theorem 3.1.8 is isonatural.*

We will use the following technical lemma.

Lemma 3.1.10. *Let X be a scheme, and U an open subscheme such that the inclusion $\iota : U \rightarrow X$ is quasi-compact. Let $f : T \rightarrow X$ be a flat morphism.*

- (1) If U is schematically dense in X then $f^{-1}(U)$ is schematically dense in T .
- (2) If f is faithfully flat, then U is schematically dense in X if and only if $f^{-1}(U)$ is schematically dense in T .

Proof. Because ι is quasi-compact and separated, we have that $\iota_*\mathcal{O}_U$ is quasi-coherent on \mathcal{O}_X , so schematic density of U in X is equivalent to injectivity of $\mathcal{O}_X \rightarrow \iota_*\mathcal{O}_U$. Similarly, $f^{-1}(U)$ is schematically dense in T if and only if $\mathcal{O}_T \rightarrow \iota_{T*}\mathcal{O}_{f^{-1}(U)}$ is injective, where $\iota_T : f^{-1}(U) \rightarrow T$ is the natural inclusion. Now suppose f is flat. The commutativity of pushforward with flat base change shows that ι_T is the pullback of ι . Thus, if ι is injective then so is ι_T (as f is flat). Moreover, if f is faithfully flat then ι is injective if and only if ι_T is injective. This establishes the lemma. \square

Remark 3.1.11. The arguments in the proof of Lemma 3.1.10 also show that an open substack $\mathcal{U} \rightarrow \mathcal{X}$ of an Artin stack is schematically dense if and only if its preimage in any smooth cover of \mathcal{X} is schematically dense.

Proof of Proposition 3.1.9. We first wish to see that being a bald Artin stack is an isonatural property. That being an Artin stack is isonatural follows from Corollary 2.1.3. Moreover, it is clear from Lemma 3.1.10 that if \mathcal{X} is an Artin stack, then the clean locus is schematically dense if and only if for some (and hence any) smooth cover $U \rightarrow \mathcal{X}$ by a scheme U , the preimage of the clean locus is schematically dense. But the preimage of the clean locus in U can be tested on the level of functors, since by Corollary 2.7.7 a point $t : T \rightarrow \mathcal{X}$ (which we will choose to factor through U) factors through the clean locus if and only if $\text{Bin}(t)$ is isomorphic to the singleton functor on the category of T -schemes. Thus we conclude that baldness is isonatural.

Next, we recall that Corollary 2.1.3 tells us that we can recognize gerbes over a given algebraic space X from their associated functors, with X recovered as the sheafification of the functor. Because X is an algebraic space determined by $F_{\mathcal{X}}$, we can impose any conditions we wish on it, so we see for instance that being a gerbe over a strongly R1 algebraic space is an isonatural property. We next note that being a neutral gerbe is isonatural, since neutrality is equivalent to having a global section.

We also remark that the gerbe classes (2)-(5) all consist of Artin stacks which are locally of finite presentation over X . Both of these properties are isonatural by Corollary 2.1.3 and Theorem 2.1.2 respectively.

For (2), we already know that being of the form BG is isonatural. We also know that G is étale if and only if any section $X \rightarrow \mathcal{X}$ is étale, which is isonatural (by the criterion of Proposition 2.4.3, applied to étale-local affines on X). Finally, G is finite if and only if \mathcal{X} is separated over X (as then the diagonal of BG – whose pullback to X is just G – is proper), which is isonatural by Theorem 2.1.2 because X is assumed to be locally Noetherian, and we are assuming that \mathcal{X} is locally of finite presentation over X .

For (3), we note that if \mathcal{X} is a neutral gerbe with global section having automorphism group sheaf G , then $\mathcal{X} \cong BG$. Thus, to test whether $\mathcal{X} \cong BG$ with G abelian, it suffices to find a global section having abelian automorphism sheaf, which is an isonatural property by Theorem 2.1.4.

The property that a gerbe \mathcal{X} is Brauer is isonatural, since \mathcal{X} is Brauer if and only if $F_{\mathcal{X}}(P)$ is non-empty for some Brauer-Severi space P over X .

Finally, we can test whether a given gerbe is a G -gerbe for G diagonalizable or equal to G_m by first verifying that all automorphism groups are abelian, and then using Corollary 2.7.15 to recover the automorphism sheaf. \square

Assuming Theorem 3.1.8, we can finish the proof that common moduli problems are isonatural.

Proof of Theorem 1.2.1. Indeed, the moduli stacks of marked curves are bald (as $\overline{\mathcal{M}}_g$ is smooth and geometrically connected [4], and the generic curve has trivial automorphism group), the Picard stack and the stacks of stable vector bundles are Brauer G_m gerbes (see Remark 3.1.12 below), while the stacks of coherent sheaves with fixed determinant and of n th roots of an invertible sheaf are Brauer μ_n gerbes over strongly R1 algebraic spaces (Theorem 9.3.3 and Theorem 9.4.3 of [13] for stable sheaves, clear for roots of an invertible sheaf). Thus, all cases follow from Theorem 3.1.8. \square

Remark 3.1.12. Let $X \rightarrow S$ be a flat projective morphism of finite presentation which is cohomologically flat in degree 0. The stack \mathcal{S}_r of stable sheaves of rank r on X is a G_m -gerbe over a separated algebraic

space S_r . Langer's work [16] shows that the connected components of \mathcal{S}_r are quasi-compact. Choose such a component Σ of \mathcal{S}_r , with sheafification (i.e., image in S_r) $\overline{\Sigma}$. Let \mathcal{V} be the universal family on $X \times \Sigma$. For sufficiently large n the sheaf $(\mathrm{pr}_2)_* \mathcal{V}(n)$ is a locally free Σ -twisted sheaf (in the terminology of [18]). Taking its projectivization yields a Brauer-Severi space $P \rightarrow \overline{\Sigma}$ with Brauer class equal to the class of Σ . This shows that $\mathcal{S}_r \rightarrow S_r$ is a Brauer G_m -gerbe. Fixing the determinant yields a Brauer μ_r -gerbe.

3.2. Bald stacks. We now show that bald Artin stacks are determined by their associated functors; that is, we prove Theorem 3.1.8(1). We make use of the formalism of groupoids in this section; this is clearly described in paragraph 2.4.3 of [17] and section 2 of [14].

Remark 3.2.1. There is a mild precursor to the main result of this section. Vistoli's proof of Proposition 2.8 of [24] implies that a tame regular separated Deligne-Mumford stack \mathcal{X} locally of finite type over a locally Noetherian algebraic space with trivial automorphism groups in codimension 1 is determined *among stacks of this form* by its coarse moduli space. Our more general result arises from the extra information made available by the associated functor.

Definition 3.2.2. Let $R \rightrightarrows Z$ be a groupoid object in the category of algebraic spaces, \mathcal{Y} a stack, and F a functor. Write $p, q : R \rightarrow Z$ for the two structure maps.

- (1) An *R -equivariant object of \mathcal{Y} over Z* is a pair (ϕ, α) with $\phi : Z \rightarrow \mathcal{Y}$ and $\alpha : \phi p \xrightarrow{\sim} \phi q$ an isomorphism of arrows $R \rightarrow \mathcal{Y}$, such that the coboundary $\delta\alpha$ equals id on $R \times_{p, Z, q} R$.
- (2) An *R -invariant object of F over Z* is a map $\psi : Z \rightarrow F$ such that $\psi p = \psi q : R \rightarrow F$.

Remark 3.2.3. We remind the reader of the definition of the coboundary $\delta\alpha$. The groupoid structure yields three maps $R \times_Z R \rightarrow R$: the two projections $\mathrm{pr}_1, \mathrm{pr}_2$ and the multiplication map m . We then have $\delta\alpha = \mathrm{pr}_1^*(\alpha) m^*(\alpha)^{-1} \mathrm{pr}_2^*(\alpha)$. Setting the coboundary equal to the identity is the same as requiring that the multiplication in R correspond to composition of arrows.

It is clear that any R -equivariant object of \mathcal{Y} over Z yields an R -invariant object of $F_{\mathcal{Y}}$ over Z . We will show that when \mathcal{Y} is bald, there is an equivalence between these two notions as long as Z dominates the clean locus and the groupoid structure maps p and q are flat.

Given a groupoid $R \rightrightarrows Z$ as above, let $[Z/R]$ be the stackification of the category fibered in groupoids whose fiber over T is the groupoid $R(T) \rightrightarrows Z(T)$. We make no assumptions on the structure maps of $R \rightrightarrows Z$, but we assume that $[Z/R]$ is the stackification in the big fppf site of the base scheme.

The R -equivariant objects of \mathcal{Y} over Z form a groupoid, in which the isomorphisms $(\phi, \alpha) \xrightarrow{\sim} (\phi', \alpha')$ are given by isomorphisms $\phi \xrightarrow{\sim} \phi'$ which are compatible with α and α' in the obvious way.

A proof of the following proposition may be found in Proposition 3.2 of [15].

Proposition 3.2.4. *The map sending $f : [Z/R] \rightarrow \mathcal{Y}$ to the associated R -equivariant object of \mathcal{Y} over Z is an equivalence of categories.*

Notation 3.2.5. Given Z, R, \mathcal{Y}, F as above, we will write \mathcal{Y}_Z^R for the set of isomorphism classes of R -equivariant objects of \mathcal{Y} over Z and F_Z^R for the set of R -invariant objects of F over Z .

It is a standard result (see Corollary 8.1.1, [17]) that $\mathrm{cl}(\mathcal{X})$ (see Definition 3.1.4 above for the definition of cl) is isomorphic to an algebraic space when \mathcal{X} is an Artin stack. Thus, suppose \mathcal{X} is a quasi-algebraic stack and $\mathcal{U} \hookrightarrow \mathcal{X}$ is a quasi-compact open immersion from an algebraic space.

Proposition 3.2.6. *Given a flat groupoid of algebraic spaces $R \rightrightarrows Z$, the functor F defines a bijection between the set of elements $[(\phi, \alpha)] \in \mathcal{X}_Z^R$ such that $\phi^{-1}(\mathcal{U}) \subseteq Z$ is schematically dense and the set of elements $\psi \in (F_{\mathcal{X}})_Z^R$ such that $\psi^{-1}(F_{\mathcal{U}}) \subseteq Z$ is schematically dense.*

Proof. Given an R -invariant object ψ of $F_{\mathcal{X}}$ over Z , choose a lift to $\phi : Z \rightarrow \mathcal{X}$. Since ψ is R -invariant, there is some isomorphism $\alpha : \phi p \xrightarrow{\sim} \phi q$. If $\psi^{-1}(F_{\mathcal{U}})$ is schematically dense in Z , then because $R \times_Z R \rightarrow Z$ is flat, it follows from Lemma 3.1.10 that $W := p^{-1}\psi^{-1}(F_{\mathcal{U}})$ is schematically dense. Since the inertia stack of \mathcal{X} is trivial, we thus see that $\delta\alpha|_W = \mathrm{id}$, and since the diagonal of \mathcal{X} is separated this implies that $\delta\alpha = \mathrm{id}$ on $R \times_Z R$. Thus, the natural map is surjective. To see that it is injective, suppose (ϕ_i, α_i) , $i = 1, 2$ are R -equivariant objects of \mathcal{X} over Z such that $\phi_i^{-1}(\mathcal{U})$ is schematically dense. If their images

in $(F_{\mathcal{X}})_Z^R$ are equal, there is some isomorphism $\beta : \phi_1 \xrightarrow{\sim} \phi_2$. It follows immediately from the schematic density hypothesis that the diagram

$$\begin{array}{ccc} \phi_1 p & \xrightarrow{\alpha_1} & \phi_1 q \\ \downarrow & & \downarrow \\ \phi_2 p & \xrightarrow{\alpha_2} & \phi_2 q \end{array}$$

commutes, which means that β is an isomorphism of R -equivariant objects. \square

Proof of Theorem 3.1.8(1). Given two bald Artin stacks \mathcal{X}_1 and \mathcal{X}_2 , let $R_i \rightrightarrows Z_i$ be a smooth presentation of \mathcal{X}_i , $i = 1, 2$. Given an isomorphism $\phi : F_{\mathcal{X}_1} \xrightarrow{\sim} F_{\mathcal{X}_2}$, there results an R_1 -invariant object of $F_{\mathcal{X}_2}$ over Z_1 . Moreover, this object must come from a smooth surjection $Z_1 \rightarrow \mathcal{X}_2$. Applying Proposition 3.2.6, there results a unique map (up to isomorphism) $\psi : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ giving rise to ϕ . Reversing the roles of 1 and 2 yields $\bar{\psi} : \mathcal{X}_2 \rightarrow \mathcal{X}_1$ inducing ϕ^{-1} . The composition $\bar{\psi}\psi : \mathcal{X}_1 \rightarrow \mathcal{X}_1$ corresponds to the R_1 -invariant object of $F_{\mathcal{X}_1}$ over Z_1 , hence must be an isomorphism. Reversing the roles of 1 and 2 again, we see that ψ is an isomorphism. That the association $\phi \mapsto \psi$ yields a retraction $\text{Isom}(F_{\mathcal{X}_1}, F_{\mathcal{X}_2}) \rightarrow \text{Isom}(\mathcal{X}_1, \mathcal{X}_2)$ follows from the uniqueness in Proposition 3.2.6 and is left to the reader. \square

3.3. Classifying stacks. In this section we show that given an algebraic space X and a finite étale group space $G \rightarrow X$, the group G is uniquely determined up to inner forms by the functor associated to the classifying stack BG . (For the reader familiar with Giraud's terminology, this says that the functor associated to BG uniquely determines the isomorphism class of the band associated to G .) This will ultimately show that classifying stacks for finite étale group spaces are isonatural.

We will recover BG by (in essence) recovering the functor of points of G (in the stack of bands) from a subcategory of pointed schemes. This subcategory arises from a functorial construction for a pointed scheme with a given finite fundamental group. (There are of course subtleties associated to doing this over schemes which are not geometric points.) We thus begin with some results pertaining to the étale fundamental group.

Lemma 3.3.1. *Suppose X is an algebraic space, $Y_1, Y_2 \rightarrow X$ are finite étale morphisms, and $f : P \rightarrow X$ is a faithfully flat morphism with geometrically connected fibers. Pullback induces a bijection between X -morphisms $Y_1 \rightarrow Y_2$ and P -morphisms $Y_1 \times_X P \rightarrow Y_2 \times_X P$.*

Proof. First suppose $Y_1 \rightarrow X$ and $Y_2 \rightarrow X$ are disjoint unions of copies of X and X is connected. Since the fibers of $P \rightarrow X$ are geometrically connected, it follows that any X -morphism $(Y_1)_P \rightarrow Y_2$ factors through a morphism $Y_1 \rightarrow Y_2$. Since $Y_1 \rightarrow X$ and $Y_2 \rightarrow X$ have this form étale locally on X , we see that the natural map of étale sheaves $\chi : \mathcal{H}om_X(Y_1, Y_2) \rightarrow f_* \mathcal{H}om_P((Y_1)_P, (Y_2)_P)$ is an epimorphism. On the other hand, since $P \rightarrow X$ is faithfully flat, χ is also a monomorphism (cf. Theorem VIII.5.2 of [11]). Thus, χ is an isomorphism, and the result follows. \square

Remark 3.3.2. Lemma 3.3.1 applies notably when $Y_1 = X$, and thus to sections of a given finite étale covering.

Proposition 3.3.3. *Suppose Z is a normal connected algebraic space and $Y \rightarrow Z$ is a smooth surjective map of finite presentation between algebraic spaces with connected geometric fibers. Let $*$: $\text{Spec } \kappa \rightarrow Y$ be a geometric point over the generic point θ of Z . Then the natural sequence of groups*

$$\pi_1(Y_{\bar{\theta}}, *) \rightarrow \pi_1(Y, *) \rightarrow \pi_1(Z, *) \rightarrow 1$$

is exact.

Proof. We may first reduce to the case that Z is excellent, using standard limiting arguments. This statement is known when Z is the spectrum of a field (Theorem IX.6.1 of [11]). Thus, as $\pi_1(Y_{\bar{\theta}}, *) \rightarrow \pi_1(Y, *)$ is surjective (Proposition V.8.2 of [ibid.]), it follows that the left-hand map in the sequence has normal image. To show that the sequence is exact in the middle, it suffices to show that any Galois cover $W \rightarrow Y$ which is trivial on the geometric generic fiber comes by pullback from Z . Since this is already known over the function field of Z , we find a field extension $L/\kappa(Z)$ such that the normalization

of Y_θ in $L|_{Y_\theta}$ is isomorphic to W_θ . Since W_θ is the generic fiber of W and Z (and thus Y) is normal, we see that the normalization of Y in $L|_{Y_\theta}$ is isomorphic to W . But since $Y \rightarrow Z$ is smooth, this is just the pullback of the normalization of Z in L . Writing $Z' \rightarrow Z$ for this normalization, we thus have that $Z' \times_Z Y \rightarrow Y$ is étale, from which it follows that $Z' \rightarrow Z$ is étale. Thus, $W \rightarrow Y$ is the pullback of an étale (in fact, Galois) covering of Z . It follows from Proposition V.6.11 of [ibid.] that the sequence is exact in the middle. Exactness on the right follows from the fact that the fibers of Y are geometrically connected (Corollary IX.5.6 of [ibid.]). (Cf. the proof of Theorem X.1.3 and Corollary X.1.4 of [ibid.].) \square

Corollary 3.3.4. *Let $f : Y \rightarrow X$ be a smooth surjective morphism of finite presentation between connected algebraic spaces with geometrically connected fibers. Let $y \rightarrow Y$ be a geometric point of Y . If $\pi_1(Y_{f(y)}, f(y)) = 0$ then the natural map $\pi_1(Y, y) \rightarrow \pi_1(X, f(y))$ is an isomorphism.*

Proof. By standard methods, we may assume that X is excellent and Noetherian; thus, the normalization $X' \rightarrow X$ is a finite surjective morphism of finite presentation. Let $W \rightarrow Y$ be a finite étale morphism. By Theorem IX.4.7 of [11], we know that $X' \rightarrow X$ is a morphism of effective descent for finite étale covers. Applying Proposition 3.3.3, we see that there is a finite étale cover $W' \rightarrow X'$ and an isomorphism $W \times_X X' \xrightarrow{\sim} W' \times_X Y$ over $Y \times_X X'$. The descent datum on W thus gives rise to a descent datum on $W' \times_X Y$ (with respect to the morphism $Y \times_X X' \rightarrow Y$). By Lemma 3.3.1, there is an induced descent datum on W' (with respect to the morphism $X' \rightarrow X$). Since $X' \rightarrow X$ is effective, there is a finite étale covering $U \rightarrow X$ giving rise to the descent datum on W' . Applying Lemma 3.3.1 once more, we see that $U \times_X Y$ and $W \times_X X'$ are isomorphic via an isomorphism preserving the descent data. Applying the effectivity to $Y \times_X X' \rightarrow Y$ once again, we see that there is an isomorphism $U \times_X Y \xrightarrow{\sim} W$.

This shows that the Galois categories of finite étale covers of Y and X (with the fiber functors induced by the given points) are equivalent, which implies that the fundamental groups are naturally isomorphic. \square

Proposition 3.3.5. *Suppose $f : Y \rightarrow X$ is a smooth surjective morphism of finite presentation with connected geometric fibers between connected algebraic spaces. Suppose $y \rightarrow Y$ is a geometric point. Let G be a finite group with a free X -action on Y . If $\pi_1(Y_{f(y)}, y) = 0$, then there is a natural isomorphism $\pi_1(Y/G, y) \xrightarrow{\sim} \pi_1(X, f(y)) \times G$.*

Proof. The Galois covering $Y \rightarrow Y/G$ induces an exact sequence

$$1 \rightarrow \pi_1(Y, y) \rightarrow \pi_1(Y/G, y) \rightarrow G \rightarrow 1.$$

By Corollary 3.3.4, the natural morphism $\pi_1(Y, y) \rightarrow \pi_1(X, f(y))$ is an isomorphism. But then the natural map $\pi_1(Y/G, y) \rightarrow \pi_1(X, f(y))$ yields a splitting of the left-hand map of the exact sequence, which yields a splitting $\pi_1(Y/G, y) \xrightarrow{\sim} \pi_1(Y, y) \times G$, as required. \square

Given a scheme S , the category $S\text{-Sch}_\bullet$ of *pointed S -schemes* is the category of S -schemes under S (i.e., arrows $S \rightarrow X$ in the category of S -schemes). Any functor $F : S\text{-Sch}^\circ \rightarrow \text{Set}$ naturally yields a functor $F_\bullet : S\text{-Sch}_\bullet \rightarrow F(S)\text{-Set}$. We will freely use the associated functors F_\bullet in studying reconstruction of stacks; since these associated functors arise abstractly from the original functor F , no additional information is introduced in their formation.

Proposition 3.3.6. *There is a functor $\beta : \text{FinGps} \rightarrow \mathbf{Z}\text{-Sch}_\bullet$ such that for any connected algebraic space X with geometric generic point $\bar{\theta} : \text{Spec } \bar{\kappa} \rightarrow X$, there is an isomorphism $\pi_1(X \times \beta(G), \bar{\theta}) \xrightarrow{\sim} \pi_1(X, \bar{\theta}) \times G$ which is functorial in G .*

To construct $\beta(G)$, we will make a functorial pointed quasi-projective \mathbf{Z} -scheme with a (functorial) free G -action. This comes in a straightforward way from the regular representation of G .

Given a finite group G , let $V(G)$ be the geometric vector bundle associated to the regular representation with its functorial A -basis $\{e_g\}_{g \in G}$. Fix a positive integer $N \geq 2$ and let $W(G) = \text{Hom}(A^N, V(G))$, where A is the trivial representation with basis 1 (so that A^N has a natural basis b_1, \dots, b_N). Let $*$ be the A -point of $W(G)$ consisting of the map sending each b_i to e_1 . The formation of $V(G)$, $W(G)$, and $*$ is clearly functorial in G and A . Let $P(G) \rightarrow \text{Spec } A$ denote the projectivization of $W(G)$. The action of G on $W(G)$ induces an action on $P(G)$, and the A -point $*$ of $W(G)$ gives rise to a natural section (which we will also denote $*$) of $P(G) \rightarrow \text{Spec } A$.

Proposition 3.3.7. *There is an open subscheme $U(G) \subset P(G)$ such that*

- (1) *$*$ is contained in $U(G)$;*
- (2) *$U(G) \rightarrow \operatorname{Spec} A$ is surjective;*
- (3) *for each geometric point $x \rightarrow \operatorname{Spec} A$, the inclusion $U(G)_x \subset P(G)_x$ is the complement of a union of hyperplanes of codimension at least 2;*
- (4) *the action of G on $U(G)$ is free;*
- (5) *given a map $\epsilon : G \rightarrow G'$ of finite groups, the induced map $W(G) \rightarrow W(G')$ induces a map $U(G) \rightarrow U(G')$ of pointed A -schemes which is ϵ -equivariant.*

Proof of Proposition 3.3.6, given Proposition 3.3.7. The proof applies immediately by applying Proposition 3.3.5 to the family $U(G) \times_{\operatorname{Spec} \mathbb{Z}} X$ given by Proposition 3.3.7 (when $A = \mathbb{Z}$). \square

We now give a proof of Proposition 3.3.7.

Lemma 3.3.8. *Let $U'(G) \subset P(G)$ be the largest open subscheme such that for any map of finite groups $G \rightarrow G'$, the rational morphism $P(G) \dashrightarrow P(G')$ is regular on $U'(G)$. Then $U'(G)$ is the complement of a union of flat families of linear subspaces of $P(G)$ of codimension at least 2.*

Proof. Any map $G \rightarrow G'$ factors through a quotient group $G \rightarrow \overline{G}$. The kernel of the induced map $W(G) \rightarrow W(\overline{G})$ is a linear space of codimension at least N (as $W(\{1\})$ has dimension N). For each such quotient $G \rightarrow \overline{G}$ there is a subbundle $W_{\overline{G}} \subset W(G)$ parametrizing the family of kernels; removing the union of the corresponding subspaces of $P(G)$ yields $U'(G)$, as desired. \square

Define a closed subscheme $Z(G) \subset U'(G)$ by taking the scheme-theoretic union of all preimages under (surjective) quotient morphisms $U'(G) \rightarrow U'(\overline{G})$ of all fixed loci for the action of non-identity elements of \overline{G} .

Lemma 3.3.9. *With the immediately preceding notation, the closed subscheme $Z(G) \subset U'(G)$ has codimension 2 in every geometric fiber of $U'(G) \rightarrow \operatorname{Spec} A$. Moreover, $*$ factors through $U'(G) \setminus Z(G)$. Finally, for every map of finite groups $G \rightarrow G'$, the induced map $U'(G) \rightarrow U'(G')$ induces a map $U'(G) \setminus Z(G) \rightarrow U'(G') \setminus Z(G')$.*

The proof of Lemma 3.3.9 requires a bit of analysis of the eigenvectors for the elements of G acting on $W(G)$.

Notation 3.3.10. Given a linear representation R of G , a scalar $\lambda \in k$, and an element $g \in G$, let $R^{g, \lambda}$ denote the submodule of R on which g acts as multiplication by λ .

If g has order ν , it is clear that $R^{g, \lambda}$ is 0 if λ is not a ν th root of unity.

Lemma 3.3.11. *With the above notation, let ν be the order of $g \in G$. Assume $g \neq 1$ (for the sake of non-stupidity). For any $\lambda \in \mu_\nu(k)$, the submodule $V(G)^{g, \lambda}$ is a locally direct summand of $V(G)$ of corank $|G|(1 - 1/\nu)$. The submodule $W(G)^{g, \lambda}$ is a locally direct summand of $W(G)$ of corank $N|G|(1 - 1/\nu) \geq N \geq 2$.*

Proof. The group decomposes into $|G|/\nu$ left orbits for the action of g of size exactly ν . Choosing an element h_i in each orbit, we see that for an element $\sum \alpha_h e_h \in V(G)^{g, \lambda}$, the coefficient of $e_{g^s h_i}$ must be $\lambda^s \alpha_{h_i}$. Thus, each element of $V(G)^{g, \lambda}$ is uniquely determined by the set of coordinates α_{h_i} , $i = 1, \dots, |G|/\nu$. This gives the statement for $V(G)$, and the assertion on $W(G)$ follows. \square

Let $f : G \rightarrow G'$ be a homomorphism of finite groups. There are induced maps $f_* : V(G) \rightarrow V(G')$ and $f_* : W(G) \rightarrow W(G')$ of A -modules which are equivariant over f in the standard sense.

Lemma 3.3.12. *Given a non-zero $\alpha \in V(G)$, suppose there is some $g' \in G'$ and $\lambda \in A$ such that $f_* \alpha \in V(G')^{g', \lambda}$. Then $g' \in f(G)$ and $f_* \alpha \in \iota_* V(f(G))^{g', \lambda}$, where $\iota : f(G) \rightarrow G'$ is the natural inclusion.*

Proof. Write $\alpha = \sum \alpha_h e_h$. By assumption, for all $h \in G$ such that $\alpha_h \neq 0$ we have that $g' f(h) \in f(G)$. Since there is some h with $\alpha_h \neq 0$, we see that $g' \in f(G)$. The lemma follows immediately. \square

Proof of Lemma 3.3.9. Since the closed subset underlying $Z(G)$ is compatible with base change, it suffices to prove the lemma assuming that $A = k$ is an algebraically closed field. A fixed point for the action of g on $U'(G)$ is the image of an eigenvector in $W(G)$. Given a quotient \overline{G} , an element $\bar{g} \in \overline{G} \setminus \{1\}$, and a scalar $\lambda \in A$, define $W(G)^{\bar{g}, \lambda}$ to be the preimage of $W(\overline{G})^{\bar{g}, \lambda}$ under the natural surjection $W(G) \rightarrow W(\overline{G})$. Considering all non-trivial quotients of G at once, we define

$$W_0(G) = W(G) \setminus \left(\bigcup_{\substack{G \rightarrow \overline{G} \neq \{1\} \\ \bar{g} \in \overline{G} \setminus \{1\}, \lambda \in k}} W(G)^{\bar{g}, \lambda} \right).$$

It is easy to see that $W_0(G)$ is the complement of the union of finitely many (locally direct summand) vector subbundles of $W(G)$ of codimension at least N and that $*$ $\in W_0(G)$. Thus, $W_0(G)$ is an open cone in $W(G)$ whose complement has codimension at least 2. Moreover, the image of $W_0(G)$ in $P(G)$ is precisely $U'(G) \setminus Z(G)$, as desired.

Applying Lemma 3.3.12, we see that given a map $G \rightarrow H$, the induced map $W(G) \rightarrow W(H)$ sends $*$ to $*$ and $W_0(G)$ into $W_0(H)$, yielding an induced pointed map $\overline{W}_0(G) \rightarrow \overline{W}_0(H)$. This gives the final functoriality statement of Lemma 3.3.9 \square

Proof of Proposition 3.3.7. Using the notation of Lemma 3.3.9, setting $U(G) = U'(G) \setminus Z(G)$ yields a functorial open subscheme with a free action (as all fixed loci have been removed) which is the complement of a union of linear subspaces of codimension at least 2 in every fiber. \square

Remark 3.3.13. The reader will note that we could have avoided the use of both the projective space $P(G)$ and the eigenspaces $W(G)^{g, \lambda}$ in characteristic 0. In that case, since $W(G)$ is itself simply connected, it suffices to simply remove the fixed point loci directly and take the quotient by G . In this guise, our construction looks more similar to that which arises in the study of equivariant cohomology, as in [5].

We next recall a few facts about bands which will be useful in the sequel. The reader is referred to Chapter IV of [7] for the definitive treatment of the subject (and further context).

Given a site S (the reader may think of the Zariski or étale site of a scheme), the stack of bands is defined as the stackification of a quotient of the stack of groups as follows: Given two sheaves of groups G and H over an object T of S , there is natural right action of $\mathcal{A}ut(G)$ (resp. left action of $\mathcal{A}ut(H)$) on $\text{Isom}(G, H)$. Define a new fibered category \overline{B} over S by taking as objects over T the set of sheaves of groups on T , but with homomorphism sheaf $\mathcal{H}om_B(G, H) = \mathcal{A}ut(H) \setminus \mathcal{H}om(G, H) / \mathcal{A}ut(G)$. The stack L_S of bands on S is then defined to be the stackification of \overline{B} .

Lemma 3.3.14. *Given an object T of S , a sheaf of groups G on T , and an inner form G' of G , there is an isomorphism $G \xrightarrow{\sim} G'$ in the category of bands.*

Proof. Since G' is an inner form of G , there is a covering $U \rightarrow T$ and an isomorphism $\phi : G|_U \xrightarrow{\sim} G'|_U$ whose coboundary, viewed as an automorphism of $G|_{U \times_T U}$, is conjugation by a section of G . But any such automorphism is trivial in the category of bands, so ϕ descends to an isomorphism $G \rightarrow G'$ in $L_S(T)$. \square

Lemma 3.3.15. *Suppose G and H are sheaves of groups on T . If $\phi : G \cong H$ in $L_S(T)$ then there is an inner form H' of H and an isomorphism of sheaves of groups $\psi : G \rightarrow H'$*

Proof. There is a covering $U \rightarrow T$ and an isomorphism $\alpha : G|_U \xrightarrow{\sim} H|_U$ whose coboundary on $U \times_T U$, viewed as an automorphism of H , is conjugation by a section $\sigma \in H(U \times_T U)$. Moreover, it is formal that σ satisfies the 1-cocycle condition, and thus yields an inner form H' of H . Composing with the natural isomorphism $H|_U \xrightarrow{\sim} H'|_U$ yields an isomorphism $G|_U \rightarrow H'|_U$ with trivial coboundary, yielding the result. \square

When S is the punctual site (e.g., the small étale site of a separably closed field), the stack of bands is just the quotient category of the category of groups which replaces $\text{Isom}(G, H)$ with the set of conjugacy classes of such isomorphisms.

Proposition 3.3.16. *Let X be a Galois category with fiber functor $*$. For any finite group G , there is a natural isomorphism between $\mathrm{Hom}_L(\pi_1(X, *), G)$ and the set of isomorphism classes of (right) G -torsors over the final object of X .*

Proof. Given a G -torsor T in the category of π -sets, the choice of a point $t \in T$ yields a homomorphism $\pi \rightarrow G$ which sends α in π to g in G such that $\alpha t = tg$. Changing the choice of t changes the map by an inner automorphism. Conversely, given such a map, one gets a left action of π on the underlying set of G which commutes with the natural right G -action. \square

Note that the functor F_{BG} (over a space X) is naturally pointed by the isomorphism class of the trivial torsor; we will use $*$ to denote the canonical point. (The reader with logical qualms should note that the fact that a stack has the form BG with G a finite étale group space can be detected from the functor by Proposition 3.1.9, and thus the pointing is isonatural.) There is a natural subfunctor F_{BG}^* on $X\text{-Sch}_\bullet$ whose value on $\sigma : X \rightarrow Y$ (where Y is an X -scheme and σ is a section of the structure morphism) is the preimage of $*$ under the restriction map $F_{BG}(Y) \rightarrow F_{BG}(X)$.

Definition 3.3.17. An isomorphism $\psi : F_{BG} \rightarrow F_{BH}$ is *pointed* if ψ sends the isomorphism class of the trivial torsor to the isomorphism class of the trivial torsor.

We will write $\mathrm{Isom}_*(F_{BG}, F_{BH})$ for the subgroup of pointed isomorphisms. It is clear that any pointed isomorphism $F_{BG} \rightarrow F_{BH}$ induces an isomorphism $F_{BG}^* \rightarrow F_{BH}^*$.

Lemma 3.3.18. *Let X be an algebraic space. Given finite groups G and H , there is a map*

$$\mathrm{Isom}_*(F_{BG}, F_{BH}) \rightarrow \mathrm{Isom}_L(G, H)$$

such that the composition $\mathrm{Isom}(G, H) \rightarrow \mathrm{Isom}_(F_{BG}, F_{BH}) \rightarrow \mathrm{Isom}_L(G, H)$ is the natural map.*

Proof. We may assume without loss of generality that X is connected. The functor $\beta : \mathrm{FinGps} \rightarrow X\text{-Sch}_\bullet$ yields a subcategory of $X\text{-Sch}_\bullet$ which is equivalent to the category of finite groups. Moreover, for any finite group Γ , we have by Proposition 3.3.16 that $F_{BG}^*(\beta(\Gamma)) \cong \mathrm{Hom}_{L_X}(\Gamma, G)$. The result thus follows from the Yoneda lemma (applied to the subcategory of bands associated to constant groups). \square

Lemma 3.3.19. *Given a section $a \in F_{BH}(X)$, there is an inner form H' of H and an isomorphism $F_{BH} \xrightarrow{\sim} F_{BH}$ carrying $*$ to a .*

Proof. If T is an H -torsor with isomorphism class a , it is standard that $H' = \mathrm{Aut}_H(T)$ is an inner form of H . Sending an H -torsor S to the H' -torsor $\mathrm{Isom}(S, T)$ gives the isomorphism in question. \square

Proposition 3.3.20. *Let G and H be finite étale group spaces over an algebraic space X . There is a map*

$$\mathrm{Isom}(F_{BG}, F_{BH}) \rightarrow \mathrm{Isom}_L(G, H)$$

whose composition with the natural map $\mathrm{Isom}(G, H) \rightarrow \mathrm{Isom}(F_{BG}, F_{BH})$ is the natural map.

Proof. There is an étale surjection $U \rightarrow X$ such that

- (1) there are finite groups \overline{G} and \overline{H} with isomorphisms $G_U \xrightarrow{\sim} \overline{G}_U$ and $H_U \xrightarrow{\sim} \overline{H}_U$;
- (2) the restriction of ψ to the category of U -schemes is pointed.

There is a resulting diagram of isomorphisms of functors on $U\text{-Sch}$

$$\begin{array}{ccc} F_{BGU} & \longrightarrow & F_{BHU} \\ \downarrow & & \downarrow \\ F_{B\overline{G}_U} & \longrightarrow & F_{B\overline{H}_U} \end{array}$$

The isomorphism $G_U \xrightarrow{\sim} \overline{G}_U$ induces a pointed automorphism of $F_{B\overline{G}}|_{U \times_X U}$ whose image in $\mathrm{Isom}_{L_{U \times_X U}}(\overline{G}, \overline{G})$ is the descent datum for G (as a form of \overline{G}); there is a similar automorphism of $F_{B\overline{H}}|_{U \times_X U}$. A straightforward (but somewhat laborious) diagram chase, starting with the global isomorphism $F_{BG} \xrightarrow{\sim} F_{BH}$, shows that the lower horizontal arrow in the above diagram respects the descent data on both sides.

Applying Lemma 3.3.19, we thus find an isomorphism $\overline{G}_U \xrightarrow{\sim} \overline{H}_U$ which is compatible with the descent data for G and H (in the stack of bands). This gives the desired map $\text{Isom}(F_{BG}, F_{BH}) \rightarrow \text{Isom}_L(G, H)$. \square

3.4. Gerbes with finite diagonalizable bands. Having treated classifying stacks, the next natural class of stacks to consider is more general non-neutral gerbes. In this section we will show Theorem 3.1.8(4): if D is diagonalizable then any Brauer D -gerbe is isonatural.

Let X be an algebraic space and A an abelian sheaf on X . In this section, the phrase “ D is a diagonalizable finite group scheme” will mean that D is isomorphic to a finite direct sum of the form $\bigoplus \mu_n$. Given an element g in a group G , write $\langle g \rangle \subseteq G$ for the cyclic subgroup generated by g .

Definition 3.4.1. Given an integer n , the *cohomology presheaf (of degree n associated to A)* is the presheaf $\mathcal{H}^n(A)$ on X -schemes such that $\mathcal{H}^n(A)(Y \rightarrow X) = H^n(Y, A_Y)$.

By common abuse of notation, we will often write simply $\mathcal{H}^n(A)(Y)$, the X -structure on Y being implicit.

Definition 3.4.2. With the above notation, given a cohomology class $\alpha \in H^n(X, A)$, the *vanishing set* of α is the set of arrows $T \rightarrow X$ such that $\alpha \in \ker(\mathcal{H}^n(A)(X) \rightarrow \mathcal{H}^n(A)(T))$. The complement of the vanishing set of α is the *support* of α . Two classes α and β have *the same support* if their supports are equal.

The sheafification of the cohomology presheaf is well-known to vanish for $n > 0$ (see for example Proposition 2.5 of Chapter II of [1]). The motivating question for this section is “How much information about a cohomology class can we recover from its support in the cohomology presheaf?” A few moments of thought will convince the reader that the best one can hope to do is recover the cyclic subgroup generated by the class, and this only in the case of a cyclic coefficient sheaf. As we will show, in various cases this actually works. However, the methods we employ are specific to the sheaves and (low) cohomological degrees in question. Further investigation of this question seems potentially interesting.

Lemma 3.4.3. *Let X be a strongly R1 algebraic space and let \mathcal{L} be an invertible sheaf on X . Let $V(\mathcal{L}) = \text{Spec Sym}^* \mathcal{L}$ be the geometric line bundle associated to \mathcal{L} . Let $Z \subseteq V(\mathcal{L})$ be the 0 section and let $V(\mathcal{L})^* = V(\mathcal{L}) \setminus Z$. Then $\text{Pic } V(\mathcal{L})^*$ is identified via pullback with $\text{Pic}(X)/\langle \mathcal{L} \rangle$.*

Proof. The hypothesis on X allows us to work with Weil divisor classes. It is well-known (with the same proof as Proposition II.6.6 of [12]) that pullback induces an isomorphism $\text{Pic}(X) \rightarrow \text{Pic}(V(\mathcal{L}))$. On the other hand, $Z \subseteq V(\mathcal{L})$ is an irreducible divisor, so $\text{Pic}(V(\mathcal{L})^*)$ is isomorphic to $\text{Pic}(V(\mathcal{L}))/\langle \mathcal{O}(Z) \rangle$. It remains to show that $\mathcal{O}(Z)|_Z \cong \mathcal{L}^\vee$ (via the natural identification of Z with X). To compute this, note that $\mathcal{O}(-Z)$ is equal to $\bigoplus_{i \geq 0} \mathcal{L}^{\otimes i}$. Restricting this to Z is the same as tensoring with $\bigoplus_{i \geq 0} \mathcal{L}^{\otimes i} / \bigoplus_{i \geq 0} \mathcal{L}^{\otimes i}$. This simply divides out by $\bigoplus_{i \geq 1} \mathcal{L}^{\otimes i}$, and thus we see that $\mathcal{O}(-Z)|_Z \cong \mathcal{L}$, as required. \square

Corollary 3.4.4. *Suppose X is a strongly R1 algebraic space. If L_1 and L_2 are invertible sheaves whose classes in $H^1(X_{\text{ét}}, \mathbf{G}_m)$ have the same support then $\langle L_1 \rangle = \langle L_2 \rangle$.*

Proof. Pulling back to $V(L_1)^*$ and using Lemma 3.4.3 and the support hypothesis, we see that $L_2 \in \langle L_1 \rangle$. Reversing the roles of L_1 and L_2 shows that $L_1 \in \langle L_2 \rangle$. The result follows. \square

Remark 3.4.5. Note that it is essential that the support of the cohomology classes be considered on the entire category of schemes and not merely on e.g. Zariski open subsets. An example is provided by $\mathcal{O}(1)$ and $\mathcal{O}(2)$ on \mathbf{P}^1 . Any scheme mapping to \mathbf{P}^1 whose image excludes a single point will trivialize both $\mathcal{O}(1)$ and $\mathcal{O}(2)$. Only by considering surjective morphisms to \mathbf{P}^1 from larger (connected) schemes can we hope to recover enough information from the support.

Lemma 3.4.6. *If $f : P \rightarrow X$ is a Brauer-Severi space then the pullback map $H^2(X, \mu_n) \rightarrow H^2(P, \mu_n)$ is injective for all n . The kernel of the map $\text{Br}(X) \rightarrow \text{Br}(P)$ is the cyclic subgroup generated by the Brauer class of P .*

Proof. The first statement follows from the fppf Leray spectral sequence for μ_n combined with the fact that $R^1 f_* \mu_n = 0$. (By the flat Kummer sequence, we know that the latter sheaf is isomorphic to the n -torsion subspace of the relative Picard space of P over X , hence vanishes.) The second statement comes from the Leray spectral sequence applied to G_m , and may be found in Theorem 2 of Part 2 of Chapter II of [6] (p. 193). \square

Lemma 3.4.7. *Let X be a strongly R1 algebraic space. If $\alpha, \beta \in H^2(X, \mu_n)$ are two Brauer cohomology classes with the same support then $\langle \alpha \rangle = \langle \beta \rangle \subseteq H^2(X, \mu_n)$.*

Proof. Since $\text{Br}(X) = \text{Br}'(X)$, there are Brauer-Severi spaces $P_\alpha \rightarrow X$ and $P_\beta \rightarrow X$ representing the images of α and β in $\text{Br}(X)$. Applying Lemma 3.4.6 to the maps $P_\alpha \times_X P_\beta \rightarrow P_\alpha \rightarrow X$, we see that it suffices to prove the lemma under the additional assumption that the Brauer classes associated to α and β are trivial. Thus, there are invertible sheaves L_α and L_β with $\alpha = c_1(L_\alpha)$ and $\beta = c_1(L_\beta)$. Using the Kummer sequence, we see that the support hypothesis is equivalent to the statement that for an algebraic space $T \rightarrow X$, the preimage $L_\alpha|_T$ is in $n \text{Pic}(T)$ if and only if $L_\beta|_T$ is in $n \text{Pic}(T)$. Let $T = \mathbf{V}(L_\alpha)^*$ as in Lemma 3.4.3, so that $\text{Pic}(T) = \text{Pic}(X)/\langle L_\alpha \rangle$. We conclude that $L_\beta \in \langle L_\alpha \rangle + n \text{Pic}(X)$. Reversing the roles of α and β , we find that $L_\alpha \in \langle L_\beta \rangle + n \text{Pic}(X)$. It follows that the images of L_α and L_β generate the same cyclic subgroup of $\text{Pic}(X)/n \text{Pic}(X)$, and this yields the result. \square

Another way to understand the statement of Lemma 3.4.7 is that there is an automorphism $\psi : \mu_n \rightarrow \mu_n$ such that $\psi^* \beta = \alpha$. In this form, the statement obviously generalizes to diagonalizable finite group schemes.

Corollary 3.4.8. *Let X be a strongly R1 algebraic space and D a diagonalizable finite group scheme. If $\alpha, \beta \in H^2(X, D)$ are Brauer cohomology classes with the same support then there is an automorphism $\psi : D \xrightarrow{\sim} D$ such that $\psi^* \beta = \alpha$.*

Proof. The proof is immediate, since D breaks up as a finite product of group schemes of the form μ_n and ψ can be defined on each factor. \square

Using Corollary 3.4.8, we will prove Theorem 3.1.8(4). It is important to note that by forgetting the automorphism data, there is no hope of recovering the gerbe structure, which consists of a specified trivialization of the inertia stack. Thus, the best we can hope for is recovery of the abstract stack, and this is indeed possible in certain situations.

Lemma 3.4.9. *Suppose X is an algebraic space and $P \rightarrow X$ is faithfully flat with geometrically connected fibers. For any finite étale group space $G \rightarrow X$, the natural map $H^1(X, G) \rightarrow H^1(P, G)$ is injective.*

Proof. By Lemma 3.3.1, given two G -torsors T and T' on X , the finite étale X -space $\text{Isom}_G(T, T')$ has a section if and only if its pullback to P has a section. The result follows. \square

Lemma 3.4.10. *Let X be a strongly R1 algebraic space and D a diagonalizable finite group scheme. Given a Brauer class $\alpha \in H^2(X, A)$, there is a faithfully flat morphism $P_\alpha \rightarrow X$ with geometrically connected fibers such that $\alpha|_{P_\alpha} = 0 \in H^2(P_\alpha, A)$.*

Proof. Writing D as a direct sum of group schemes of the form μ_n , it immediately follows that we may assume $D = \mu_n$. The class α has an image $\bar{\alpha} \in \text{Br}'(X) = \text{Br}(X)$ (since X is tasty). Let $P_0 \rightarrow X$ be a Brauer-Severi space representing $\bar{\alpha}$, so that $\alpha|_{P_0}$ is the first Chern class of an invertible sheaf $L \in \text{Pic}(P_0)$. Applying Lemma 3.4.3, we see that there is a faithfully flat map $P_\alpha \rightarrow P_0$ such that L becomes an n th power (in fact, trivial) on P_α . It follows that $\alpha|_{P_\alpha} = 0$, as required. \square

Proposition 3.4.11. *Let D be a diagonalizable finite group scheme and let \mathcal{X} be a D -gerbe over a strongly R1 algebraic space X . If \mathcal{Y} is a quasi-algebraic stack and $F_{\mathcal{X}}$ is isomorphic to $F_{\mathcal{Y}}$ then \mathcal{X} is isomorphic to \mathcal{Y} .*

Proof. Since $F_{\mathcal{X}}$ and $F_{\mathcal{Y}}$ are isomorphic, we know by Corollary 2.1.3(3) that there is a 1-morphism $\mathcal{Y} \rightarrow \mathcal{X}$ making \mathcal{Y} a gerbe. By Proposition 3.3.20 the automorphism groups at geometric points of \mathcal{Y} are all (non-canonically) isomorphic to A . It follows that the band G of \mathcal{Y}/X is a form of D_X . Since D is abelian (so that bands and groups are equivalent), we have that G is classified by an element of

$H^1(X, \text{Aut}(D))$. Since $\text{Aut}(D)$ is a finite étale group scheme, it follows from Lemma 3.3.1 that we can detect triviality of this cohomology class after pulling back along any faithfully flat morphism with geometrically connected fibers. Thus, applying Lemma 3.4.10, we may pull back to $P \rightarrow X$ so that $\mathcal{X}|_P \cong BD$. In this case, \mathcal{Y} also has a global section, via ϕ , so that $\mathcal{Y} \cong BG$. Now the triviality of the band follows from Proposition 3.3.20. We may thus choose an identification of the band of \mathcal{Y} (on X , by applying Lemma 3.3.1 to $\text{Isom}_X(G, D)$) with D .

Write $\alpha \in H^2(X, D)$ for the class corresponding to \mathcal{X} and $\beta \in H^2(X, D)$ for the class corresponding to \mathcal{Y} . Via ϕ , we see that α and β have the same support. Using the fact that α is Brauer, this then implies that β is also Brauer. By Corollary 3.4.8 there is an isomorphism $\psi : D \xrightarrow{\sim} D$ such that $\psi^*\beta = \alpha$. Composing the D -structure on \mathcal{Y} with ψ^{-1} produces a new D -gerbe structure on \mathcal{Y} for which the cohomology classes associated to \mathcal{X} and \mathcal{Y} agree. By Giraud's fundamental theorem (Theorem 3.4.2(i) of Chapter IV of [7]), there is an (D -linear) isomorphism $\mathcal{X} \rightarrow \mathcal{Y}$ over X , yielding the result. \square

3.5. \mathbf{G}_m -gerbes. Let X be an algebraic space and $\mathcal{X}_i \rightarrow X$, $i = 1, 2$ two \mathbf{G}_m -gerbes such that $[\mathcal{X}_1] \in \text{Br}(X) \subseteq H^2(X, \mathbf{G}_m)$. Write n for the order of $[\mathcal{X}_1]$. Our final task is to prove Theorem 3.1.8(5), which is the following statement.

Proposition 3.5.1. *If $F_{\mathcal{X}_1}$ and $F_{\mathcal{X}_2}$ are isomorphic then \mathcal{X}_1 and \mathcal{X}_2 are isomorphic.*

In other words, $[\mathcal{X}_1] = \pm[\mathcal{X}_2]$ in $H^2(X, \mathbf{G}_m)$.

Lemma 3.5.2. *If $\mathcal{Y} \rightarrow X$ and $\mathcal{Z} \rightarrow X$ are abelian gerbes, then there is a natural map*

$$\text{Isom}_X(F_{\mathcal{Y}}, F_{\mathcal{Z}}) \rightarrow \text{Isom}(L(\mathcal{Y}), L(\mathcal{Z})),$$

where $L(\mathcal{Y})$ and $L(\mathcal{Z})$ denote the bands of the gerbes.

Proof. This follows from the proof of Corollary 2.7.15. \square

Let $\pi : P \rightarrow X$ be a Brauer-Severi space with Brauer class $[\mathcal{X}_1]$. We can view π as a Čech covering in the fppf topology. It is elementary that $\check{H}^2(\{P \rightarrow X\}, \mathbf{G}_m) = 0$. The Čech-to-derived spectral sequence thus yields an exact sequence

$$(4) \quad 0 \rightarrow \check{H}^1(\{P \rightarrow X\}, \mathcal{H}^1(\mathbf{G}_m)) \rightarrow H^2(X, \mathbf{G}_m) \rightarrow \check{H}^0(\{P \rightarrow X\}, \mathcal{H}^2(\mathbf{G}_m)).$$

Lemma 3.5.3. *The natural map $\check{H}^0(\{P \rightarrow X\}, \mathcal{H}^2(\mathbf{G}_m)) \rightarrow H^2(P, \mathbf{G}_m)$ is an isomorphism.*

Proof. The presheaf $\mathcal{H}^2(\mathbf{G}_m)$ assigns to an X -space Y the group $H^2(Y, \mathbf{G}_m)$. We get \check{H}^0 of the presheaf by forming the equalizer of the diagram

$$H^2(P, \mathbf{G}_m) \xrightleftharpoons[\text{pr}_2^*]{\text{pr}_1^*} H^2(P \times_X P, \mathbf{G}_m).$$

The Leray spectral sequence shows that the pullback map $\pi^* : H^2(X, \mathbf{G}_m) \rightarrow H^2(P, \mathbf{G}_m)$ is surjective. Since $\pi\pi_1 = \pi\pi_2$, we see that $\text{pr}_1^* = \text{pr}_2^*$, so that the equalizer is $H^2(P, \mathbf{G}_m)$, as desired. \square

Corollary 3.5.4. *There is a natural isomorphism $\check{H}^1(\{P \rightarrow X\}, \mathcal{H}^1(\mathbf{G}_m)) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$ onto the subgroup of $H^2(X, \mathbf{G}_m)$ generated by $[\mathcal{X}_1]$.*

Proof. By Lemma 3.5.3, $\check{H}^1(\{P \rightarrow X\}, \mathcal{H}^1(\mathbf{G}_m))$ is identified (via the edge map in the spectral sequence) with the kernel of the pullback map $H^2(X, \mathbf{G}_m) \rightarrow H^2(P, \mathbf{G}_m)$. The argument cited in the proof of Lemma 3.4.6 shows that this kernel is precisely the subgroup generated by $[P] = [\mathcal{X}_1]$. \square

Let $\bar{x} \rightarrow X$ be a geometric point. There is a canonical isomorphism

$$(5) \quad \mathbf{Z} \xrightarrow{\sim} \text{Pic}(P_{\bar{x}})$$

given by the unique ample generator.

Lemma 3.5.5. *There is a natural injection*

$$\check{H}^1(\{P \rightarrow X\}, \mathcal{H}^1(\mathbf{G}_m)) \hookrightarrow (\mathbf{Z} \times \mathbf{Z})/(n, -n)$$

onto the subgroup spanned by $(1, -1)$ arising from the restriction of cocycles to the fiber of $P \times_X P$ over \bar{x} and the isomorphism of equation (5) above.

Proof. The Čech cohomology group is the cohomology group at the middle node of the diagram

$$\mathrm{Pic}(P) \rightrightarrows \mathrm{Pic}(P \times_X P) \rightrightarrows \mathrm{Pic}(P \times_X P \times_X P).$$

We know that the relative Picard space of an ℓ -fold product of P over X is \mathbf{Z}^ℓ . Sending a cohomology class to the associated section of the relative Picard space yields a diagram

$$(6) \quad \begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Pic}(X) & \rightrightarrows & \mathrm{Pic}(X) & \rightrightarrows & \mathrm{Pic}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Pic}(P) & \rightrightarrows & \mathrm{Pic}(P \times_X P) & \rightrightarrows & \mathrm{Pic}(P \times_X P \times_X P) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Z} & \rightrightarrows & \mathbf{Z}^2 & \rightrightarrows & \mathbf{Z}^3 \end{array}$$

with exact columns and whose top and bottom rows are acyclic at the middle node. Moreover, the bottom vertical maps agree with the ones given by restricting to the geometric fibers over \bar{x} . A straightforward calculation shows that the horizontal kernel at \mathbf{Z}^2 is the subgroup generated by $(1, -1)$. In addition, it is standard that the image of $\mathrm{Pic}(P)$ in \mathbf{Z} is the subgroup generated by n . This yields a map

$$\check{H}^1(\{P \rightarrow X\}, \mathcal{H}^1(\mathbf{G}_m)) \rightarrow \mathbf{Z}^2/(n, -n),$$

which we claim is an isomorphism onto the subgroup generated by $(1, -1)$. That the image lies in that subgroup follows from the preceding sentences. By Corollary 3.5.4, it is enough to show that $(1, -1)$ is in the image of $\mathrm{Pic}(P \times_X P) \rightarrow \mathbf{Z}^2$; a simple chase through diagram (6) then shows that one can find such an element which is in the horizontal kernel at $\mathrm{Pic}(P \times_X P)$.

We claim that in fact the map $\mathrm{Pic}(P \times_X P) \rightarrow \mathbf{Z}^2$ is surjective. To see this, first note that $P \times_X P \rightarrow P$ (by either projection) is a trivial Brauer-Severi space, so that there is an invertible sheaf \mathcal{L} on $P \times_X P$ which is a relative $\mathcal{O}(1)$ for such a projection. More precisely, the canonical relatively ample section 1 in \mathbf{Z}_X (the relative Picard space) lifts to a section of the Picard stack (i.e., an invertible sheaf) upon pullback to P , and it is this sheaf which we take for \mathcal{L} . It follows from this description that the restriction of \mathcal{L} to the geometric fiber over \bar{x} has class $(1, 0)$. Similarly, we can find \mathcal{M} mapping to $(0, 1)$. The claim follows, and with it the result. \square

Lemma 3.5.6. *Given an isomorphism $\gamma : F_{\mathrm{BG}_m}|_{P \times_X P} \xrightarrow{\sim} F_{\mathrm{BG}_m}|_{P \times_X P}$ such that $\gamma(P \times_X P)(*) = *$, we have that $\gamma(P_{\bar{x}} \times_{\bar{x}} P_{\bar{x}}) = \mathrm{id}$ or $-\mathrm{id}$ as automorphisms of the set $\mathbf{Z}^{\oplus 2} = F_{\mathrm{BG}_m}(P_{\bar{x}} \times_{\bar{x}} P_{\bar{x}})$.*

Proof. Choosing a section $y \in P_{\bar{x}}(\bar{x})$ yields two maps $P_{\bar{x}} \rightarrow P_{\bar{x}} \times_{\bar{x}} P_{\bar{x}}$ (identifying the fibers of the projections over y) which, by cohomology and base change, yield an isomorphism $\mathrm{Pic}(P_{\bar{x}} \times_{\bar{x}} P_{\bar{x}}) \xrightarrow{\sim} \mathrm{Pic}(P_{\bar{x}}) \times \mathrm{Pic}(P_{\bar{x}})$. Composing this with pullback along the diagonal embedding $P_{\bar{x}} \rightarrow P_{\bar{x}} \times_{\bar{x}} P_{\bar{x}}$ yields the group law on $\mathrm{Pic}(P_{\bar{x}}) = \mathbf{Z}$. Since all of the maps in question are derived from geometric constructions, they are γ -equivariant (as maps of sets). It follows using the fact that $\gamma(*) = *$ that the action of γ on $F_{\mathrm{BG}_m}(P_{\bar{x}})$ is by a group automorphism of \mathbf{Z} . Thus, γ acts on $F_{\mathrm{BG}_m}(P_{\bar{x}})$ as id or $-\mathrm{id}$. Since the proof produces a γ -equivariant isomorphism $F_{\mathrm{BG}_m}(P_{\bar{x}} \times_{\bar{x}} P_{\bar{x}}) \xrightarrow{\sim} F_{\mathrm{BG}_m}(P_{\bar{x}}) \times F_{\mathrm{BG}_m}(P_{\bar{x}})$, the result follows. \square

Corollary 3.5.7. *Let a and b be two elements of $F_{\mathcal{X}_2}(P \times_X P)$. If*

$$f, g : F_{\mathrm{BG}_m}|_{P \times_X P} \xrightarrow{\sim} F_{\mathcal{X}_2}|_{P \times_X P}$$

are two isomorphisms sending $$ to a then $f^{-1}(b)|_{(P \times_X P)_{\bar{x}}} = \pm g^{-1}(b)|_{(P \times_X P)_{\bar{x}}}$.*

Proof. Consider the diagram of isomorphisms

$$\begin{array}{ccc} F_{\mathbf{B}\mathbf{G}_m} & & \\ \downarrow \gamma & \searrow f & \\ & & F_{\mathcal{X}_2} \\ & \nearrow g & \\ F_{\mathbf{B}\mathbf{G}_m} & & \end{array}$$

where $\gamma = g^{-1}f$. By assumption $\gamma(P \times_X P)(*) = *$. We conclude by Lemma 3.5.6 that $\gamma(P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}}) = \pm \text{id}$, so that $g(P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}}) = \pm f(P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}})$. Thus, we conclude that $f^{-1}(b)|_{(P \times P)_{\overline{x}}} = \pm g^{-1}(b)|_{(P \times P)_{\overline{x}}}$. \square

Lemma 3.5.8. *An element $F_{\mathcal{X}_i}(T)$ determines a unique isomorphism $F_{\mathbf{B}\mathbf{G}_m|_T} \xrightarrow{\sim} F_{\mathcal{X}_i|_T}$ whose induced morphism of bands is $\text{id} : \mathbf{G}_m \rightarrow \mathbf{G}_m$.*

Proof. Given an invertible sheaf \mathcal{L} and an object σ of $(\mathcal{X}_i)_T$, there results a new object $\sigma \otimes \mathcal{L}$ of $(\mathcal{X}_i)_T$ by standard methods (e.g., one can use the descent datum for \mathcal{L} and the fact that \mathcal{X}_i is a \mathbf{G}_m -gerbe to produce a form of σ with the same cocycle). Moreover, replacing σ by an isomorphic object σ' yields an isomorphic object $\sigma' \otimes \mathcal{L}$, and likewise for \mathcal{L} . Finally, any object τ of $(\mathcal{X}_i)_T$ has the form $\sigma \otimes \mathcal{L}$ for a unique \mathcal{L} . These statements are also functorial in T . The result follows. \square

Lemma 3.5.9. *Let α be an element of $F_{\mathcal{X}_2}(P)$. Suppose the two pullbacks of α to $F_{\mathcal{X}_2}(P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}})$ are a and b . For any isomorphism $\phi : F_{\mathbf{B}\mathbf{G}_m \times (P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}})} \xrightarrow{\sim} F_{\mathcal{X}_2 \times (P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}})}$ sending $*$ to a , we have that the image of $\phi^{-1}(b)$ under the map*

$$\text{Pic}(P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}}) \rightarrow (\mathbf{Z} \times \mathbf{Z})/(n, -n)$$

represents either $[\mathcal{X}_2]$ or $-[\mathcal{X}_2]$ via the injection of Lemma 3.5.5.

Proof. In this proof we freely use the theory of twisted sheaves as developed in [18]. Suppose first that ϕ is the restriction of the isomorphism arising from α as in Lemma 3.5.8. We can compute $\phi^{-1}(b)$ as follows: the element α corresponds to an \mathcal{X}_2 -twisted invertible sheaf \mathcal{L} on P (up to isomorphism), and $\phi^{-1}(b)$ is the isomorphism class of the restriction of $\text{pr}_1^* \mathcal{L} \otimes \text{pr}_2^* \mathcal{L}^\vee$ to $P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}}$. As an element of $\mathbf{Z}^{\oplus 2}$, this lies in the kernel of the coboundary map in the bottom row of diagram (6) above and is thus a multiple of $(1, -1)$ (i.e., a cocycle). Furthermore, changing \mathcal{L} by an invertible sheaf on P changes the resulting cocycle by a coboundary, leaving the cohomology class (in $\check{H}^1(\{P \rightarrow X\}, \mathcal{H}^1(\mathbf{G}_m))$) invariant.

On the other hand, since \mathcal{X}_2 is trivial on P , we know by Lemma 3.4.6 that $[\mathcal{X}_2] = d[\mathcal{X}_1]$ for some d , and this implies that we can identify \mathcal{X}_2 -twisted sheaves with d -fold \mathcal{X}_1 -twisted sheaves. In particular, if \mathcal{M} is an \mathcal{X}_1 -twisted invertible sheaf on P , we may assume (for the purposes of computing the cohomology class) that $\mathcal{L} = \mathcal{M}^{\otimes d}$. But then we find that $\phi^{-1}(b)$ is d times the class of $\text{pr}_1^* \mathcal{M} \otimes \text{pr}_2^* \mathcal{M}^\vee$. Since the latter is precisely the image of $[\mathcal{X}_1]$ in $\check{H}^1(\{P \rightarrow X\}, \mathcal{H}^1(\mathbf{G}_m))$, the result then follows from Lemma 3.5.5.

When ϕ is not the isomorphism induced by α , we can apply Corollary 3.5.7 to compare the two, yielding the possible change of sign. \square

Proof of Proposition 3.5.1. Let $\psi : F_{\mathcal{X}_1} \rightarrow F_{\mathcal{X}_2}$ be an isomorphism and let $\pi : P \rightarrow X$ be a Brauer-Severi space with cohomology class $[\mathcal{X}_1]$, as above. The isomorphism ψ induces an isomorphism of bands $L_\psi : L_1 \xrightarrow{\sim} L_2$, so that \mathcal{X}_2 is also a \mathbf{G}_m -gerbe.

Since \mathcal{X}_1 is trivial on P , there is an isomorphism $F_{\mathbf{B}\mathbf{G}_m \times P} \xrightarrow{\sim} F_{\mathcal{X}_1 \times_X P}$ coming from an element $\alpha \in F_{\mathcal{X}_1}(P)$. Composing with ψ yields an isomorphism $F_{\mathbf{B}\mathbf{G}_m \times P} \rightarrow F_{\mathcal{X}_2 \times P}$ sending $*$ to $\psi(\alpha)$. By Lemma 3.5.9 (applied to the pairs $\mathcal{X}_1, \mathcal{X}_1$ and $\mathcal{X}_2, \mathcal{X}_2$), we know that the two preimages of α (resp. $\psi(\alpha)$) in $F_{\mathbf{B}\mathbf{G}_m}(P_{\overline{x}} \times_{\overline{x}} P_{\overline{x}})$ differ by a representative for the cohomology class of $[\mathcal{X}_1]$ (resp. $\pm[\mathcal{X}_2]$) in $\check{H}^1(P, \mathcal{H}^1(\mathbf{G}_m))$. By Corollary 3.5.4, we conclude that $[\mathcal{X}_1] = \pm[\mathcal{X}_2]$, as desired (as the change of sign corresponds to changing the trivialization of the band of \mathcal{X}_2 and does not change the underlying stack structure). \square

3.6. Counterexamples. On sites smaller than $S\text{-Sch}$, one can construct various examples of stacks which are not isonatural.

(1) On the small étale site of an algebraically closed field k , the stacks BG for any group G all have the singleton sheaf as associated functor. When G is finite, these stacks have representable diagonals, satisfy the kind of limiting property we require for quasi-algebraic stacks, etc. In this case, the underlying site clearly does not contain the kind of “anabelian” structures needed to reconstruct anything.

(2) If X is a geometrically unibranch scheme and F is a (discrete) torsion free abelian group, then BF again has associated functor represented by X . The diagonal is again representable, and the diagonal of BF again satisfies the desired limiting property with respect to inverse systems of objects of the small étale site of X .

(3) The small Zariski site also lacks anabelian structure: the stack BG_m has singleton associated functor on the small Zariski site of A^1 , while BG has singleton associated functor on the small Zariski site of any irreducible scheme for any discrete group G .

(4) Using the techniques developed in Section 3.5, one can also make families of examples where the associated functor is (marginally) more complicated. Let us show that as long as there is an element $\alpha \in \text{Br}(k)$ of order invertible in k and larger than 4, there are G_m -gerbes \mathcal{X} and \mathcal{Y} on the small étale site of k such that $F_{\mathcal{X}}$ is isomorphic to $F_{\mathcal{Y}}$ but \mathcal{X} is not isomorphic to \mathcal{Y} (as a stack).

There are two important sheaves on $k_{\text{ét}}$: the sheaf of multiplicative groups G_m and the sheaf μ_{∞} of all roots of unity. It is clear that restriction defines a natural map $\mathcal{A}ut(G_m) \rightarrow \mathcal{A}ut(\mu_{\infty})$. Kummer theory shows that the natural inclusion $\mu_{\infty} \hookrightarrow G_m$ induces an isomorphism $H^2(k_{\text{ét}}, \mu_{\infty})' \xrightarrow{\sim} H^2(k_{\text{ét}}, G_m)'$ (where the superscripts indicate the prime-to-characteristic parts of the groups in question).

Lemma 3.6.1. *The natural map $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathcal{A}ut(\mu_{\infty})$ sending 1 to inversion is an isomorphism.*

Proof. With respect to any chosen (non-canonical) isomorphism $\mu_{\infty}(\bar{k}) \xrightarrow{\sim} \widehat{\mathbb{Z}}$, the sheaf of automorphisms gets identified with continuous Galois-equivariant automorphisms of $\widehat{\mathbb{Z}}$. But the continuous automorphism group of $\widehat{\mathbb{Z}}$ is $\mathbb{Z}/2\mathbb{Z}$, generated by inversion. Since inversion is clearly Galois-equivariant, the result follows. \square

Corollary 3.6.2. *The orbits of the action of $\mathcal{A}ut(G_m)$ on $\text{Br}(k)'$ are given by $\{\alpha, -\alpha\}$ for $\alpha \in \text{Br}(k)'$.*

Proof. Because every automorphism of G_m induces an automorphism of μ_{∞} compatibly with the respective actions on cohomology, the corollary follows from Lemma 3.6.1 and the fact that $H^2(k_{\text{ét}}, \mu_{\infty})' \rightarrow H^2(k_{\text{ét}}, G_m)'$ is an isomorphism. \square

Given an element $\alpha \in \text{Br}(k)$, write $\langle \alpha \rangle$ for the subgroup generated by α .

Proposition 3.6.3. *Suppose $\alpha \in \text{Br}(k)'$ has order larger than 4, so that the generators for $\langle \alpha \rangle$ lie in at least two orbits under the automorphism group of G_m . Then there are two non-isomorphic stacks \mathcal{X} and \mathcal{Y} such that $F_{\mathcal{X}}$ and $F_{\mathcal{Y}}$ are isomorphic.*

Proof. Let β be a generator for $\langle \alpha \rangle$ which is distinct from α and $-\alpha$. Let \mathcal{X} be a G_m -gerbe representing α and \mathcal{Y} a G_m -gerbe representing β . It is elementary that the support of α and β in $k_{\text{ét}}$ are the same. On the other hand, if $F_{\mathcal{X}}(L) \neq \emptyset$ then it is a singleton (since two sections of BG_m differ by an invertible sheaf, of which there is only one on $L_{\text{ét}}$), which means that $F_{\mathcal{X}}$ and $F_{\mathcal{Y}}$ are (even canonically!) isomorphic.

If $\mathcal{X} \xrightarrow{\sim} \mathcal{Y}$ is an isomorphism, then it induces an isomorphism of the bands. Changing the trivialization of the band of \mathcal{Y} by an automorphism of G_m yields two G_m -gerbes with the same cohomology class. But we know from Corollary 3.6.2 that the orbit of β for the automorphism group of G_m is $\{\beta, -\beta\}$. Since α is neither β nor $-\beta$, we see that there cannot be an isomorphism between \mathcal{X} and \mathcal{Y} . \square

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